Measuring risk concentration

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We consider the following stochastic credit portfolio loss model:

$$L = \sum_{i=1}^{n} L_i. \tag{1}$$

$L_1, \ldots, L_n \geq 0$ are random variables that represent the losses that a financial institution suffers on its exposures to borrowers $i = 1, \ldots, n$ within a fixed time-period, e.g. one year. The random variable L then expresses the portfolio-wide loss. We denote by $\mathrm{P}[\ldots]$ the real-world probability distribution that underlies model (1). In other words, $\mathrm{P}[\ldots]$ is calibrated in such a way that it reflects as close as possible observed loss frequencies.

It is common practice for financial institutions to measure the risk inherent in their portfolios in terms of economic capital (EC). As credit risk, for most institutions, is considered to be most important, this is in particular relevant for credit portfolios. EC is commonly understood as a capital buffer intended to cover the losses of the lending financial institution with a high probability. This interpretation makes appear very natural the definition

$$\mathrm{EC} = \mathrm{VaR}_{\mathrm{P},\alpha}(L) - \mathrm{E}_{\mathrm{P}}[L], \tag{2}$$

where the *Value-at-Risk* (VaR) is given as a high-level (e.g. $\alpha = 99.9\%$) quantile of the portfolio-wide loss:

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L) = \min\{\ell : \mathrm{P}[L \leq \ell] \geq \alpha\}. \tag{3}$$

Hence, if a financial institutions holds EC according to (2) and charges the loans granted with upfront fees adding up to $\mathrm{E}_{\mathrm{P}}[L]$, the probability that it will lose all its EC is not higher than $1 - \alpha$.

Active risk management involves more than just measuring portfolio-wide capital according to (2). Additionally, it is of interest to identify which parts of the portfolio bind the largest portions of EC. The corresponding process of determining a risk-sensitive decomposition of EC is called *capital allocation.* While for the expectation part $\mathrm{E}_{\mathrm{P}}[L]$ of EC on the right-hand side of (2) there is the natural decomposition

$$\mathrm{E}_{\mathrm{P}}[L] = \sum_{i=1}^{n} \mathrm{E}_{\mathrm{P}}[L_i], \tag{4}$$

there is no such obvious decomposition

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L) = \sum_{i=1}^{n} \mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) \tag{5}$$

for the VaR-part of EC into *risk contributions.* Interpreting risk sensitivity as compatibility with portfolio optimization, [5] proved that the risk contributions

$\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ on the right-hand side of (5) should be defined as directional derivatives, i.e.

$$\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) \;=\; \frac{d\,\mathrm{VaR}_{\mathrm{P},\alpha}(L + h\,L_i)}{d\,h}\Big|_{h=0}. \tag{6}$$

As VaR is a positively homogeneous risk measure, by Euler's theorem, then (5) holds. Additionally, it turns out [2, 5] that, under fairly general conditions on the joint distribution of L and L_i, the derivative (6) coincides with an expectation of the loss related to borrower i conditional on the event of observing a portfolio-wide loss equal to VaR.

$$\frac{d\,\mathrm{VaR}_{\mathrm{P},\alpha}(L + h\,L_i)}{d\,h}\Big|_{h=0} \;=\; \mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)] \tag{7}$$

As an important application of the risk contribution concept (6), [6] proposed to use it for measuring *risk concentration* and diversification, in the following sense:

Definition 1 *Let $L_1, \ldots, L_n$ be loss variables and let $L = \sum_{i=1}^n L_i$. Then*

$$\mathrm{DI}_{\mathrm{P},\alpha}(L) \;=\; \frac{\mathrm{VaR}_{\mathrm{P},\alpha}(L) - \mathrm{E}_{\mathrm{P}}[L]}{\sum_{i=1}^n \mathrm{VaR}_{\mathrm{P},\alpha}(L_i) - \mathrm{E}_{\mathrm{P}}[L]}$$

denotes the diversification index *of portfolio L with respect to EC based on* $\mathrm{VaR}_{\mathrm{P},\alpha}$. *The fraction*

$$\mathrm{DI}_{\mathrm{P},\alpha}(L_i \,|\, L) \;=\; \frac{\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L) - \mathrm{E}_{\mathrm{P}}[L_i]}{\mathrm{VaR}_{\mathrm{P},\alpha}(L_i) - \mathrm{E}_{\mathrm{P}}[L_i]}$$

denotes the marginal diversification index *of sub-portfolio L_i with respect to EC based on* $\mathrm{VaR}_{\mathrm{P},\alpha}$.

In general, $\mathrm{DI}_{\mathrm{P},\alpha}(L)$ assuming a value close to 1 will indicate that there is no significant diversification in the portfolio. Similarly, a value close to 1 of $\mathrm{DI}_{\mathrm{P},\alpha}(L_i \,|\, L)$ will indicate that there is almost no diversification effect with credit i. As the dependence − measured as degree of comonotonicity − in a portfolio is influenced both by idiosyncratic and systematic risk, the diversification indices according to Definition 1 capture name diversification as well as sectoral diversification.

In general, no closed-form representations of $\mathrm{VaR}_{\mathrm{P},\alpha}(L)$ and the risk contributions $\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ are available. Therefore, often, these quantities can only be inferred from Monte-Carlo samples. This means essentially to generate a sample

$$(L^{(t)}, L_1^{(t)}, \ldots, L_n^{(t)}), \quad t = 1, \ldots, T, \tag{8}$$

and then to estimate the quantities under consideration on the basis of this sample. How to do this is quite obvious for VaR, but is much less clear for the risk contributions $\mathrm{VaR}_{\mathrm{P},\alpha}(L_i \,|\, L)$ as, in general, estimating derivatives of stochastic quantities without closed-form representation is a subtle issue. If $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is positive, the conditional expectation on the right-hand side of (7) is given by

$$\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)] \;=\; \frac{\mathrm{E}_{\mathrm{P}}[L_i\,\mathbf{1}_{\{L=\mathrm{VaR}_{\mathrm{P},\alpha}(L)\}}]}{\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]}. \tag{9}$$

Even if $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is positive, its magnitude will usually be very small, such as $1-\alpha$ or less. [1] showed how to apply *importance sampling* in such a situation in order to efficiently estimate $\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$. [3] and [4] applied similar techniques to the problem of estimating contributions to Expected Shortfall.

However, a crucial condition for (7) to hold exactly is the existence of a density of the distribution of L. The probability $\mathrm{P}[L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ then equals zero, and consequently the right-hand side of (9) is undefined. In this situation, the conditional expectation $\mathrm{E}_{\mathrm{P}}[L_i \,|\, L = \mathrm{VaR}_{\mathrm{P},\alpha}(L)]$ is still well-defined by the Radon-Nikodym theorem, but its estimation from a sample like (8) requires more elaborated non-parametric methods. We follow here [2] who applied *kernel estimation* methods for VaR contributions when optimizing returns in a portfolio of stocks. The kernel estimation procedures, however, have to be adapted to the rare-event character of credit risk. Therefore, in [7] we modify the approach by [2] in a way that can be described as a combination of kernel estimation and importance sampling.

So far, the approach proposed in [7] is not yet fully satisfactory. In particular, the following issues are open:

- Finding an efficient way of choosing an optimal or nearly optimal tilting parameter for the exponential tilting procedure.
- Can the estimation performance be improved by using other kernel estimators than the Nadaraya-Watson estimator?
- How to adapt efficiently the approach by [3] to shifting the distribution of the systematic factors for estimating VaR contributions?

References

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