Validation of internal rating systems and PD estimates

Dirk Tasche*†

May 2006

1 Introduction

This chapter elaborates on the validation requirements for rating systems and probabilities of default (PDs) which were introduced with the New Capital Standards (commonly called "Basel II", cf. BCBS, 2004). We start in Section 2 with some introductory remarks on the topics and approaches that will be discussed later on. Then we have a view on the developments in banking regulation that have enforced the interest of the public in validation techniques. When doing so, we put the main emphasis on the issues with *quantitative validation*. The techniques discussed here could be used in order to meet the quantitative regulatory requirements. However, their appropriateness will depend on the specific conditions under which they are applied.

In order to have a common ground for the description of the different techniques, we introduce in Section 3 a theoretical framework that will be the basis for the further considerations. Intuitively, a good rating system should show higher probabilities of default for the less creditworthy rating grades. Therefore, in Section 4, we discuss how this monotonicity property is reflected in the theoretical framework from Section 3.

In Section 5, we study the meaning of *discriminatory power* and some tools for measuring it in some detail. We will see that there are tools that might be more appropriate than others for the purpose of regulatory validation of discriminatory power. The topic in Section 6 is *calibration* of rating systems. We introduce some of the tests that can be used for checking correct calibration and discuss the properties of the different tests. We then conclude in Section 7 with some comments on the question which tools might be most appropriate for quantitative validation of rating systems and probabilities of default.

2 Regulatory background

There is a long tradition of rating agencies grading firms that issue bonds. This aims primarily at facilitating the decision making of investors. Very roughly, the rating methodology applied

*Deutsche Bundesbank, Postfach 10 06 02, 60006 Frankfurt am Main, Germany
E-mail: dirk.tasche@gmx.net

†The opinions expressed in this paper are those of the author and do not necessarily reflect views of Deutsche Bundesbank.

by the agencies could be decribed as expert judgment that is based on hard as well as on soft facts.

Credit institutions have another fifty years long tradition of scoring borrowers. This way, the credit institutions want to support credit decisions, i.e. decisions to grant credit or not. With regard to scoring, the predominant methodology applied by the credit institutions could roughly be described as use of statistically based score variables.

In the past, rating and scoring were regarded as being rather different concepts. This was partly caused by the fact that rating and scoring respectively are usually applied to populations with rather different characteristics. Ratings are most frequently used for pricing of bonds issued by larger corporates. Scores variables are primarily used for retail credit granting.

But also the background of the developers of rating methodologies and scoring methodologies respectively is usually quite different. Rating systems are often developed by experienced practitioners, whereas the development of score variables tends to be conferred on experts in statistics.

With the rising of modern credit risk management, a more unified view of rating and scoring has become common. This is related to the fact that today both rating and score systems are primarily used for determining PDs of borrowers. PDs are the crucial determinants for pricing and granting credit as well as for allocating regulatory and internal capital for credit risks.

In BCBS (2004), the Basel Committee on Banking Supervision recommends to take rating and scoring as the basis for determining risk-sensitive regulatory capital requirements for credit risks (Basel II). Compared to the Basel I standard, where capital requirements are uniformly at eight percent in particular for corporate borrowers irrespective of their creditworthiness, this is a major progress.

Credit institutions that apply the Basel II standardized approach can base the calculation of capital requirements on agency ratings which are called *external ratings* in the Basel II wording. However, at least in continental Europe, external ratings are available only for a minority of the corporate borrowers. As a consequence, in practice the capital requirements according to the standardized will not differ much from the requirements according to the Basel I regime.

Credit institutions that are allowed to apply the internal ratings based (IRB) approach will have to derive PDs from ratings or scores they have determined themselves. Such ratings or scores are called *internal ratings*. The PDs then are the main determinants of the regulatory capital requirements. Note that in the IRB approach capital requirements depend not only on PD estimates but also on estimates of LGD (loss given default) and EAD (exposure at default) parameters. Validation of LGD and EAD estimates is not a topic in this chapter.

As mentioned earlier, there are different ways to develop internal rating systems. On the one hand, there is the traditional approach to rating which is primarily based on expert knowledge. The number of rating grades is fixed in advance and assignments of grades are carried out according to qualitative descriptions of the grades in terms of economic strength and creditworthiness.

On the other hand, another – also more or less traditional approach – is scoring which is primarily based on statistical methods. The first result then is a score variable that takes on values on a continuous scale or in a discrete range with many possible outcomes. The Basel II IRB approach requires that the score values are then mapped on a relatively small number of rating grades (at least seven non-default grades), but leaves the exact number of grades in the institution's discretion.

Combinations of rating systems that are based on statistical models and rating systems that are based on expert knowledge are called *hybrid* models. All kinds of combinations appear in practice, with quite different combination approaches. Driven partly by an IRB approach requirement, hybrid models even seem to be predominant. Often they occur in the shape of a statistical model whose output can be overridden by expert decisions.

Among the rules on validation in the Basel II framework, two are particularly relevant for statistically based quantitative validation (see BCBS, 2004, § 500).

- "Banks must have a robust system in place to validate the accuracy and consistency of rating systems, processes, and the *estimation of all relevant risk components*."
- "A bank must demonstrate to its supervisor that the internal validation process enables it to assess the *performance of internal rating* and risk estimation systems consistently and meaningfully."

The Basel Committee on Banking Supervision has established the Accord Implementation Group (AIG) as a body where supervisors exchange minds on implementation questions and provide general principles for the implementation of the Basel II framework. In particular, the AIG has proposed general principles for validation. Most of these principles are related to the validation process as such, and only some are relevant for quantitative validation. In the following list of principles (cf. BCBS, 2005a) the ones relevant for quantitative validation are emphasized.

(i) Validation is fundamentally about assessing the *predictive ability of a bank's risk estimates* and the use of ratings in credit processes.

(ii) The bank has primary responsibility for validation.

(iii) Validation is an iterative process.

(iv) There is *no single validation method*.

(v) Validation should encompass both *quantitative* and qualitative elements.

(vi) Validation processes and outcomes should be subject to independent review.

Hence, in particular, the Basel Committee emphasizes that validation is not only a quantitative statistical issue, but also involves an important qualitative process-oriented component. This qualitative component of validation is commonly considered equally if not more important than the quantitative component. This chapter, however, deals with the quantitative component of validation only[1].

Principle (i) of the AIG introduces the term "predictive ability". This is not a common statistical notion. It is not a priori clear whether it is related to well-known technical terms like "unbiasedness", "consistency" and so on. However, there seems to be a consensus in the financial industry that "predictive ability" should be understood in terms of *discriminatory power* and correctness of *calibration* of rating systems. We follow this path of interpretation for the rest of the chapter.

[1]For more information on qualitative validation see, e.g., CEBS (2005) .

Commonly[2], discriminatory power is considered to be related to the discrimination between "good" and "bad" borrowers. Additionally, there is a connotation of discriminatory power with the correctness of the ranking of the borrowers by the rating system. While the importance of discriminatory power is obvious, however, examining the ranking seems to be of secondary importance, as in the end the ranking should be according to the size of the PD estimates. Therefore, correct ranking will be reached as soon as the calibration of the rating system is correct. This is a consequence of the fact that correct calibration is usually understood as having found the "true" PDs (probabilities of default) for the rating grades. Correctness of the calibration of a rating system may be understood as implementation of the Basel Committee's requirement to assess the quality of the *estimation of all relevant risk components*. Checking discriminatory power may be interpreted as implementation of the Basel Committee's requirement to validate the *performance of internal rating*.

With regard to quantitative validation, the Basel Committee states in § 501 of BCBS (2004) "Banks must regularly *compare realised default rates with estimated PDs for each grade* and be able to demonstrate that the realised default rates are within the expected range for that grade." Hence there is a need for the institutions to compare PD estimates and realized default rates at the level of single rating grades. Such a procedure is commonly called *back-testing*.

In § 502 the committee declares "Banks must *also use other quantitative validation tools* and comparisons with relevant external data sources". Thus, institutions are required to think about further validation methods besides back-testing at grade-level.

In § 504 of BCBS (2004) the Basel Committee requires that "Banks must have well-articulated internal standards for situations where *deviations in realised PDs, LGDs and EADs from expectations become significant* enough to call the validity of the estimates into question. These standards must take account of *business cycles and similar systematic variability* in default experiences." As a consequence, institutions have to decide whether perceived differences of estimates and realized values are really significant. Additionally, the committee expects that validation methods take account of systematic dependence in the data samples used for estimating the risk parameters PD, LGD and EAD.

In the retail exposure class, institutions need not apply fully-fledged rating systems in order to determine PDs for their borrowers. Instead, they may assign borrowers to pools according to similar risk characteristics. Obviously, the requirements for quantitative validation introduced so far have to be modified accordingly.

3 Statistical background

Conceptual considerations. The goal with this section on the statistical background is to introduce the model which will serve as the unifying framework for most of the more technical considerations in the part on validation techniques. We begin with some conceptual considerations.

We look at rating systems in a *binary classification* framework. In particular, we will show that the binary classification concept is compatible with the idea of having more than two rating grades. For the purpose of this chapter binary classification is understood in the sense of discriminating between the populations of defaulters and non-defaulters respectively.

[2]In this respect, we follow BCBS (2005b).

For the purpose of this chapter we assume that the score or rating grade S (based on regression or other methods) assigned to a borrower summarizes the information which is contained in a set of co-variates (e.g. accounting ratios). Rating or score variable design, development or implementation[3] is not the topic of this presentation. We want to judge with statistical methods whether rating or scoring systems are appropriate for discrimination between "good" and "bad" and are well calibrated.

With regard to calibration, at the end of the section we briefly discuss how PDs can be derived from the distributions of the scores in the population of the defaulters and non-defaulters respectively.

Basic setting. We assume that with every borrower two random variables are associated. There is a variable S that may take on values across the whole spectrum of real numbers. And there is another variable Z that takes on the values D and N only. The variable S denotes a score on a continuous scale that the institution has assigned to the borrower. It thus reflects the institutions's assessment of the borrower's creditworthiness. We restrict our considerations to the case of continuous scores since this facilitates notation and reasoning. The case of scores or ratings with values in a discrete spectrum can be treated much in a similar way. The variable Z shows the state the borrower will have at the end of a fixed time-period, say after one year. This state can be *default*, D, or *non-default*, N. Of course, the borrower's state in a year is not known today. Therefore, Z is a *latent* variable.

The institutions's intention with the score variable S is to forecast the borrower's future state Z, by relying on the information on the borrower's creditworthiness that is summarized in S. In this sense, scoring and rating are related to binary classification. Technically speaking, scoring can be called binary classification with a one-dimensional co-variate.

Describing the joint distribution of $(\mathbf{S}, \mathbf{Z})$ with conditional densities. As we intend statistical inference on the connections between the score variable S and the default state variable Z, we need some information about the joint statistical distribution of S and Z. One way to describe this joint distribution is by specifying first the marginal distribution of Z and then the conditional distribution of S given values of Z.

Keep in mind that the population of borrowers we are considering here is composed by the sub-population of future defaulters, characterized by the value D of the state variable Z, and the sub-population of borrowers remaining solvent in the future, characterized by the value N of the state variable Z. Hence, borrowers with $Z = D$ belong to the defaulters population, borrowers with $Z = N$ to the non-defaulters population.

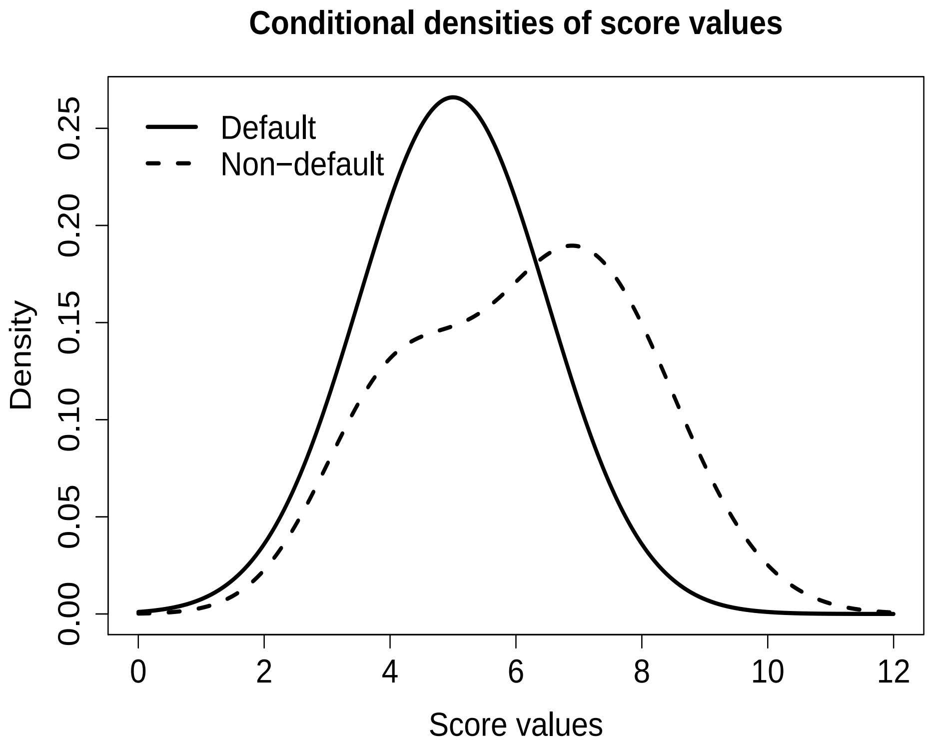
The marginal distribution of Z is very simple as it suffices to specify p, the *total probability of default* (also called *unconditional PD* in the whole population. Hence p is the probability that the state variable Z takes on the value D. It also equals 1 minus the probability that Z takes on the value N.

$$p \;=\; \mathrm{P}[Z = D] \;=\; 1 - \mathrm{P}[Z = N]. \tag{3.1a}$$

Note that the conditional probability of default given that the state variable takes on D is just 1 whereas the conditional PD given that the state is N is just 0. As the score variable S is continuous by assumption, its conditional distributions given the two values Z can take on may

[3] The process of design and implementation should be subject to qualitative validation.

Figure 1: *Illustrative example of score densities conditional on the borrower's status (default or non-default).*



be specified by *conditional densities* f_D and f_N respectively. Figure 1 illustrates how a plot of such conditional densities might look like. The probability that the score S is not greater than some value s given that the state variable Z equals – say – D can be expressed as an integral of the density f_D.

$$F_z(s) \;=\; \mathrm{P}[S \leq s \,|\, Z = z] \;=\; \int_{-\infty}^{s} f_z(u)\, du, \quad z = D, N \tag{3.1b}$$

Describing the joint distribution of $(\mathbf{S}, \mathbf{Z})$ with conditional probabilities of default. Another, in a sense dual way of describing the joint distribution of S and Z is to specify for every value the score variable can take on the conditional probability $\mathrm{P}[Z = D \,|\, S = s]$ that the state variable Z equals D. This is nothing but the conditional PD given the score. See Figure 2 for an example of how the graph of such a conditional PD function could look like.

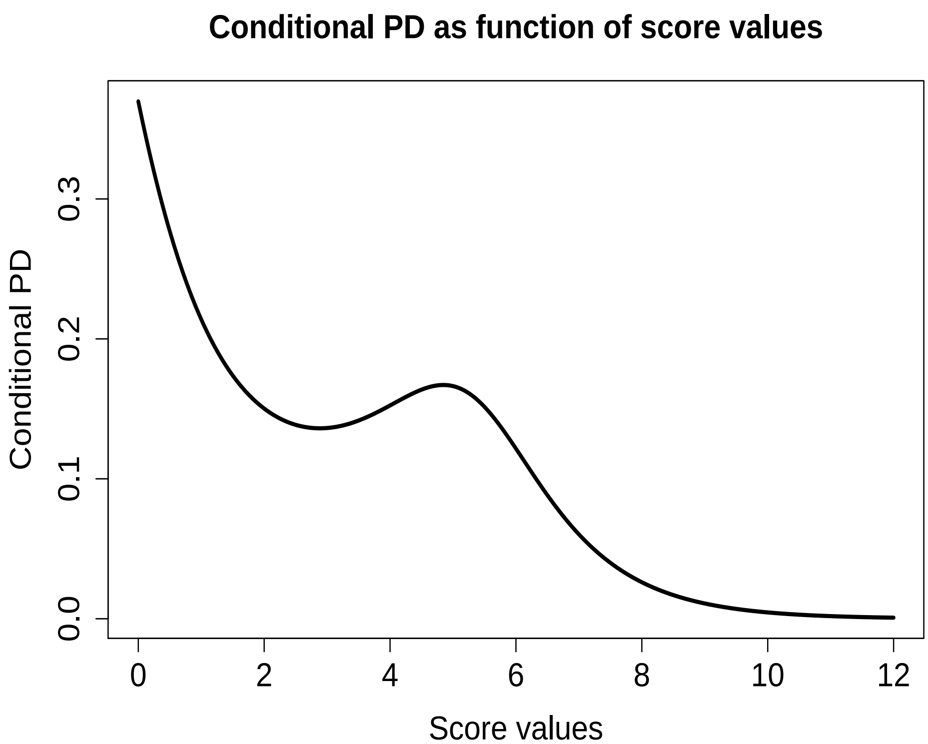
$$\begin{aligned} s \;\mapsto\; \mathrm{P}[Z = D \,|\, S = s] &= \mathrm{P}[D \,|\, S = s] \\ &= 1 - \mathrm{P}[N \,|\, S = s]. \end{aligned} \tag{3.2a}$$

In order to fully specify the joint distribution, in a second step then the unconditional distribution of the score variable S has to be fixed, e.g. via an unconditional density f.

$$\mathrm{P}[S \leq s] \;=\; \int_{-\infty}^{s} f(u)\, du. \tag{3.2b}$$

Note the difference between the unconditional density f of the score variable S on the one hand and the on the state variable Z conditioned densities f_D and f_N on the other hand. The unconditional density f gives the distribution of the scores in the whole population, whereas f_D and f_N describe the score distribution on sub-populations only. If the score variable S really bears information about the default state, then the three densities will indeed be different. If not, the densities might be similar or even identical. By means of the densities f, f_D and

Figure 2: *Illustrative example of PD conditional on score values. Calculated with the densities from Figure 1 according to* (3.3b). *Total PD 10 percent.*



f_N the distributions of all, the "defaulting" and the "non-defaulting" respectively borrowers' score-variables are determined.

Equivalence of the both descriptions. So far, we have seen two descriptions of the joint distribution of the score variable and the state variable which are quite different at first glance. However, thanks to Bayes' formula both descriptions are actually equivalent.

Suppose first that a description of the joint distribution of score and state by the total probability of default p and the two conditional densities f_D and f_N according to (3.1a) and (3.1b) is known. Then the unconditional score density f can be expressed as

$$f(s) = p\, f_D(s) + (1-p)\, f_N(s), \tag{3.3a}$$

and the conditional PD given that the score variable takes on the value s can be written as

$$\mathrm{P}[D \,|\, S = s] = \frac{p\, f_D(s)}{f(s)}. \tag{3.3b}$$

Assume now that the unconditional score density f in the sense of (3.2b) and the function representing the conditional PDs given the scores in the sense of (3.2a) are known. Then the total PD p can be calculated as an integral of the unconditional density f and the conditional PD as

$$p = \int_{-\infty}^{\infty} \mathrm{P}[D \,|\, S = s]\, f(s)\, ds, \tag{3.4a}$$

and the both conditional densities of the score variable can be obtained via

$$\begin{aligned} f_D(s) &= \mathrm{P}[D \,|\, S = s]\, f(s)/p \quad \text{and} \\ f_N(s) &= \mathrm{P}[N \,|\, S = s]\, f(s)/(1-p). \end{aligned} \tag{3.4b}$$

A comment on conditional PDs. It is useful to keep in mind that (3.3b) is only one of several ways to calculate conditional PDs. By definition, the conditional PD $\mathrm{P}[D \,|\, S]$ can also be described as the best forecast of the default/non-default state variable Z by a function of S in the least squares sense, i.e.

$$\mathrm{P}[D \,|\, S] = \arg \min_{Y=f(S),\ f \text{ function}} \mathrm{E}\big[(Z-Y)^2\big]. \tag{3.5}$$

This means that the conditional PD can be regarded as the solution of an optimization problem where the objective is to approximate as best as possible the state variable by some function of the score variable. Or, alternatively, it can be stated that given the information by the score S, there is no better approximation (in the least squares sense) of the state variable Z than $\mathrm{P}[D \,|\, S]$. Intuitively this is quite clear, because obviously a conditional PD of – say – 90 percent would indicate that the borrower under consideration is close to default.

Note that due to the continuous distribution of S for any $s \in \mathbb{R}$ we have $\mathrm{P}[S = s] = 0$. Hence $\mathrm{P}[D \,|\, S]$ is a conditional probability in the non-elementary sense and has to be dealt with carefully.

Dealing with cyclical effects. Recall from Section 2 that, for modelling, estimation and validation, institutions have to take account of "business cycles and similar systematic variability". We consider here how such cyclical effects, expressed as additional dependence on time, can be incorporated in the model framework we have introduced.

A first way to incorporate time-dependence is to assume that the conditional PDs $\mathrm{P}[D \,|\, S = \cdot]$ provided by the model are constant over time and to admit time-dependence only via the unconditional score density f, i.e. f is replaced by $f(\cdot, t)$. This corresponds to the so-called *through-the-cycle* (TTC) rating philosophy, where rating grades are assumed to express the same degree of creditworthiness at any time and economic downturns are only reflected by a shift of the score distribution towards the worse scores.

A second possibility to take account of time-dependence could be to assume that the conditional PDs are varying with time, i.e.

$$\mathrm{P}[D \,|\, S = \cdot] = \mathrm{P}_t[D \,|\, S = \cdot]. \tag{3.6}$$

This can be modelled with an assumption of constant conditional score densities f_D and f_N, having only the total PD $p = p_t$ time-dependent. Via Bayes' formula (3.3b), the conditional PDs would then depend upon time too. This approach corresponds to the *point-in-time* (PIT) rating philosophy according to which one and the same rating grade can reflect different degrees of creditworthiness, depending on the state of the economy.

The situation in practice. If we think in terms of a statistically based rating or scoring system, we can expect that from the development of the score variable conditional densities of the scores in the two populations of defaulters and non-defaulters respectively are known. In some institutions, it is then practice to predict the total probability of default for the following period of time by means of a regression on macro-economic data. In such a situation, Bayes' formula is useful for deriving the conditional PDs given the values of the score variable. Note that we are then in the point-in-time context we have described above. In some cases, for instance when logit or probit regression is applied, the score variable itself can be interpreted as conditional PD. This would again be a case of a point-in-time rating philosophy.

The popular software by Moody's-KMV provides a further example of this type. There one can also say that the score variable is identical with the conditional probability of default. The traditional Moody's (or S&P or Fitch) ratings are commonly considered to be examples of the through-the cycle rating philosophy.

Mapping score values on rating grades. Due to our assumptions and hence relevant for many score variables in practice, the theoretical probability that the score variable S takes on some fixed score value s is 0. As a consequence of this fact PDs conditional on single scores s cannot directly be back-tested since there will not be many or even no observations of borrowers with score s. In order to facilitate validation, therefore, the Basel Committee on Banking Supervision decided to admit for the IRB approach only rating systems with a finite number of grades. Thus, if a rating system is based on a continuous score variable, a reasonable mapping of the scores on the grades must be constructed. In the remaining part of this section, we show how such a mapping can be constructed while taking into account the intended PDs of the grades.

The main issue with mapping score values on rating grades is to fix the criterion according to which the mapping is defined. We consider two different quantitative criteria. The first criterion for the mapping we consider is the requirement to have *constant PDs over time*.

To describe the details of the corresponding mapping exercise, assume that k non-default rating grades have to be defined. Grade 1 denotes the grade that indicates the highest creditworthiness. Increasing PDs $q_1 < \ldots < q_{k-1}$ have been fixed. Assume additionally, that the conditional PD $\mathrm{P}[D \,|\, S = s]$ as a function of the score values is decreasing in its argument s. We will come back to a justification of this assumption in Section 4.

Given the model framework introduced above, the theoretical conditional PD given that the score variable takes on a value equal to or higher than a fixed limit s_1 can be determined. This observation also holds for the PD conditional on the event that the score variable takes on a value between two fixed limits. Speaking technically, it is possible, by proceeding recursively, to find limits $s_1 > s_2 > \ldots > s_{k-1}$ such that

$$\begin{aligned} q_1 &= \mathrm{P}[D \,|\, S \geq s_1] = \frac{p\int_{s_1}^{\infty} f_D(u)\,du}{\int_{s_1}^{\infty} f(u)\,du} \quad \text{and} \\ q_i &= \mathrm{P}[D \,|\, s_{i-1} > S \geq s_i] = \frac{p\int_{s_i}^{s_{i-1}} f_D(u)\,du}{\int_{s_i}^{s_{i-1}} f(u)\,du} \quad \text{for } i = 2, \ldots, k-1. \end{aligned} \tag{3.7a}$$

If a borrower has been assigned a score value s equal to or higher than the limit s_1, he or she receives the best grade $R(s) = 1$. In general, if the score value is less than a limit s_{i-1} but equal to or higher than the next limit s_i then the intermediate grade $R(s) = i$ is assigned. Finally, if the borrower's score value is less than the lowest limit s_{k-1} then he or she receives the worst grade $R(s) = k$.

$$R(s) \;=\; \begin{cases} 1 & \text{if } s \geq s_1, \\ i & \text{if } s_{i-1} > s \geq s_i,\ i = 2, \ldots, k-1, \\ k & \text{if } s_{k-1} > s. \end{cases} \tag{3.7b}$$

The PD of the worst grade k cannot be fixed in advance. It turns out that its value is completely determined by the PDs of the better grades. This value, the conditional PD given that the rating grade is k, can be calculated in a way similar to the calculations of the conditional PDs for the better grades in (3.7a). As the result we obtain

$$\mathrm{P}[D \,|\, R(S) = k] \;=\; \mathrm{P}[D \,|\, s_{k-1} > S] \;=\; p\,\frac{\int_{-\infty}^{s_{k-1}} f_D(u)\,du}{\int_{-\infty}^{s_{k-1}} f(u)\,du}. \tag{3.7c}$$

As $s \mapsto \mathrm{P}[D \,|\, S = s]$ decreases, $r \mapsto \mathrm{P}[D \,|\, R(S) = r]$ is an increasing function, as should be expected intuitively.

The second criterion for the mapping we consider is the requirement to have a *constant distribution of the good borrowers across the grades over time*. The intention with such a criterion might be to avoid major shifts with respect to ratings in the portfolio when the economy undergoes a downturn.

Assume hence that a number k of non-default grades (grade 1 best) and shares $0 < r_1, \ldots, r_k$, $\sum_{i=1}^k r_i = 1$, of good borrowers in the grades have been fixed. Assume again additionally that high score values indicate high creditworthiness. It turns out that it is then possible, again by proceeding recursively, to find limits $s_1 > s_2 > \ldots > s_{k-1}$ such that

$$\begin{aligned} r_1 &= \mathrm{P}[S \geq s_1 \,|\, N] = \int_{s_1}^{\infty} f_N(s)\,ds, \\ r_i &= \mathrm{P}[s_{i-1} > S \geq s_i \,|\, N] = \int_{s_i}^{s_{i-1}} f_N(s)\,ds,\ i = 2, \ldots, k-1. \end{aligned} \tag{3.8a}$$

The mapping of the scores on grades in this case is again defined by (3.7b). Since $\sum_{i=1}^k r_i = 1$ this definition implies immediately

$$\mathrm{P}[R(S) = k \,|\, N] \;=\; \mathrm{P}[S < s_{k-1} \,|\, N] \;=\; r_k. \tag{3.8b}$$

Note that in both cases of mapping criteria we have considered, in order to keep constant over time the PDs and the shares of good borrowers respectively, the limits $s_1 \ldots, s_{k-1}$ must be periodically updated.

4 Monotonicity of conditional PDs

The mapping procedures described in the last part of Section 3 work under the assumption that the conditional PD given the score $\mathrm{P}[D \,|\, S = s]$ is a function that decreases in its argument s. As Figure 2 demonstrates this need not be the case. Are there any reasonable conditions such that monotonicity of the conditional PDs given the score is guaranteed?

An answer[4] to this question, in particular, will provide us with a justification of the monotonicity assumption which underlies Equations (3.7a) to (3.7c). This assumption is needed to ensure that the proposed mapping procedure for having constant PDs over time really works.

We discuss the question in the context of a *hypothetical decision problem*. Assume that we consider a borrower chosen at random and have been informed about his or her score value. But, of course, we do not yet know whether the borrower will default or not. How could we infer the value of his or her default state variable Z? In formal terms: suppose that a realization (s, z) of (S, Z) has been sampled. s is observed, z is not yet visible. Is $z = N$ or $z = D$?

One way to come to a decision would be to fix some set A of score values such that we infer default, D, as state if the borrower's score value is in A. If the value of the score were not in A, we would conclude that the state is N for non-default, i.e.

$$\begin{aligned} s \in A \;&\Rightarrow\; \text{Conclusion } z = D \\ s \notin A \;&\Rightarrow\; \text{Conclusion } z = N. \end{aligned} \tag{4.1}$$

How should we choose the *acceptance set* A? A convenient way to deal with the problem of how to find A is to have recourse to statistical test theory.

Then the problem can be stated as having to discriminate between the conditional score distributions $\mathrm{P}[S \in \cdot \,|\, D]$ on the defaulters and $\mathrm{P}[S \in \cdot \,|\, N]$ on the non-defaulters sub-populations respectively. Thus we have to decide whether the borrower under consideration stems from the defaulters population or from the non-defaulters population. The key concept for solving this problem is to have as objective a high certainty in the case of a decision to reject the presumption that the borrower is a future defaulter.

Formally, we can state this concept as follows: the null hypothesis is that the borrower is a future defaulter, or, equivalently, that his or her state variable takes on the value D. The alternative hypothesis is that the borrower's state variable has got the value N.

- *Null hypothesis:* $\mathrm{P}[S \in \cdot \,|\, D]$, i.e. $z = D$.
- *Alternative:* $\mathrm{P}[S \in \cdot \,|\, N]$, i.e. $z = N$.

We conduct a statistical test on the null hypothesis "state equals D" against the alternative "state equals N". Hence, our decision could be wrong in two ways. The so-called type I error would be to reject "state equals D" although the state is actually D. The so-called type II error would be to accept "state equals D" although "state equals N" is true.

- *Type I error*: erroneously rejecting $z = D$.
- *Type II error*: erroneously accepting $z = D$.

In order to arrive at an optimal decision criterion, the probabilities of the two possible erroneous decisions have to be considered: The probability of the type I error is the probability under the defaulters' score distribution that a borrower's score will *not* be an element of the acceptance set A.

$$\mathrm{P}[\text{Type I error}] \;=\; \mathrm{P}[S \notin A \,|\, D]. \tag{4.2a}$$

[4] This section is based on Tasche (2002).

In contrast, the probability of the type II error is the probability under the non-defaulters' score distribution that a borrower's score will be an element of the acceptance set A.

$$\mathrm{P}[\text{Type II error}] \;=\; \mathrm{P}[S \in A \,|\, N]. \tag{4.2b}$$

The type I error probability is usually limited from above by a small constant. Common values are 1 or 5 percent, but we will see in Section 5 that for the purpose of validation also higher values make sense. Having bounded the type I error probability from above, the objective is to minimize the type II error probability.

$$\begin{gathered} \mathrm{P}[\text{Type I error}] \leq \text{ (small) constant} \\ \mathrm{P}[\text{Type II error}] \text{ as small as possible.} \end{gathered} \tag{4.3}$$

Note that $1 - \mathrm{P}[\text{Type II error}] = \mathrm{P}[S \notin A \,|\, N]$ is called the *power of the test* we are conducting.

The optimal solution for the decision problem (4.3) is provided by the well-known Neyman-Pearson lemma. We state here briefly a slightly simplified version. See Casella and Berger (2001) or other textbooks on statistics for a more detailed version of the lemma. Let now α denote a fixed bound for the type I error probability, say α equal to 5 percent. Such an α is called *confidence level*.

The first step in stating the *Neyman-Pearson lemma* is to introduce another random variable, the so-called *likelihood ratio*. It is obtained by applying the ratio of the non-defaulters and the defaulters conditional score densities f_N and f_D respectively as a function to the score variable itself. The second step is to determine the $1-\alpha$-quantile r_α of the likelihood ratio.

$$r_\alpha \;=\; \min\big\{r \geq 0 :\, \mathrm{P}\big[\frac{f_N}{f_D}(S) \leq r \,\big|\, D\big] \geq 1-\alpha\big\}. \tag{4.4a}$$

In the case of any reasonable assumption on the nature of the conditional continuous score distributions this can be done by equating $1-\alpha$ and the probability that the likelihood ratio is not higher than the quantile, i.e.

$$1-\alpha \;=\; \mathrm{P}\big[\frac{f_N}{f_D}(S) \leq r_\alpha \,\big|\, D\big], \tag{4.4b}$$

and then solving the equation for the quantile r_α. Having found the quantile of the likelihood ratio, the decision rule "Reject the hypothesis that the future state is D if the likelihood ratio is greater than the quantile" is optimal among all the decision rules that guarantee a type I error probability not greater than α. Hence, formally stated, the decision rule

$$\frac{f_N}{f_D}(S) \;>\; r_\alpha \qquad \Longleftrightarrow \qquad \text{rejecting } D \tag{4.5a}$$

minimizes the type II error under the condition

$$\mathrm{P}[\text{Type I error}] \;\leq\; \alpha. \tag{4.5b}$$

As a consequence, for any decision rule of the shape

$$S \notin A \qquad \Longleftrightarrow \qquad \text{rejecting } D \tag{4.6a}$$

and with $\mathrm{P}[S \notin A \,|\, D] \le \alpha$ we have

$$\mathrm{P}\big[\frac{f_N}{f_D}(S) \le r_\alpha \,\big|\, N\big] \;\le\; \mathrm{P}[S \in A \,|\, N]. \tag{4.6b}$$

In other words, any optimal test of D ("defaulter") against N ("non-defaulter") at level α looks like the likelihood ratio test, i.e. from (4.6b) follows

$$A \;=\; \big\{s : \frac{f_N(s)}{f_D(s)} \le r_\alpha\big\}. \tag{4.7}$$

Actually, (4.7) is not only the optimal decision criterion for discriminating between defaulters and non-defaulters, but also provides an answer to the original question of when the conditional PD given the scores is a monotonous function.

Cut-off decision rules. To explain this relation we need a further definition. A score variable S is called to be of *cut-off type* with respect to the distributions $\mathrm{P}[S \in \cdot \,|\, D]$ and $\mathrm{P}[S \in \cdot \,|\, N]$, if for every type I error probability α a decision rule of half-line shape

$$S > r_\alpha \qquad \Longleftrightarrow \qquad \text{rejecting } D \tag{4.8a}$$

or for every α a rule of half-line shape

$$S < r_\alpha \qquad \Longleftrightarrow \qquad \text{rejecting } D \tag{4.8b}$$

is *optimal* in the sense of minimizing the type II error probability under the constraint (4.5b). Decision rules as in (4.8a) and (4.8b) are called *cut-off rules*.

By (4.7), we can now conclude that

the score variable S is of cut-off type with respect to $\mathrm{P}[S \in \cdot \,|\, D]$ and $\mathrm{P}[S \in \cdot \,|\, N]$, if and only if the likelihood ratio $s \mapsto f_N(s)/f_D(s)$ is monotonous.

Note that, for any score variable, its corresponding likelihood ratio is of cut-off type.

Conclusions for practical applications. Bayes' formula (3.3b) shows that the likelihood ratio is monotonous if and only if the conditional PD $s \mapsto \mathrm{P}[D \,|\, S = s]$ is monotonous. There are some theoretical examples where the likelihood ration is indeed monotonous: for instance when both conditional densities f_N and f_D are normal densities, with equal standard deviation.

Unfortunately, in practice, monotonicity of the likelihood ratio or the conditional PD is hard to verify. However, from economic considerations can be clear that cut-off decision rules for detecting potential defaulters are optimal. This may justify the assumption of monotonicity. If, however, non-monotonicity of the likelihood ratio is visible from graphs as in Figure 2, the reliability of the score variable may be questioned. This yields a first example for a validation criterion for score variables, namely is the likelihood ratio monotonous or not?

5 Discriminatory power of rating systems

The following section is devoted to studying the question of how discriminatory power can be measured and tested. We have seen in Section 3 that the statistical properties of a score variable can to a high extent be expressed by the conditional densities of the score variable on the two populations of the defaulters and non-defaulters respectively. Another, closely related way of characterization is by means of the conditional probability of default given the score values. With this observation in mind, discriminatory power can roughly be described in technical terms as discrepancy of the conditional densities, as variation of the conditional PD or as having the conditional PDs as close as possible to 100 percent or 0 percent.

Many statistical tools are available for measuring discriminatory power in one of these ways. We will consider a selection of tools that enjoy some popularity in the industry. A first major differentiation among the tools can be applied according to whether or not their use involves estimation of the total (or portfolio-wide or unconditional) probability of default. If this is necessary, the tool can be applied only to samples with the right proportion of defaulters. If estimation of the total PD is not involved, the tool can also be applied to non-representative samples. This may be important in particular when the power shall be estimated on the development sample of a rating system. The presentation of tools in the remaining part of this section closely follows the presentation in Chapter III of BCBS (2005b). Of course, the presented list of tools for measuring discriminatory power is not exhaustive. Which tool should be preferred may strongly depend on the intended application. As a consequence, various scientific disciplines like statistics in medicine, signal theory, or weather forecasting in the course of time suggested quite different approaches on how to measure the discriminatory power of classification systems.

Cumulative Accuracy Profile (CAP). The Cumulative Accuracy Profile (or CAP) is a useful graphical tool for investigating the discriminatory power of rating systems. Recall from (3.1b) the notions F_N and F_D for the distribution functions of the score variable on the non-defaulters' and defaulters' respectively populations. Denoting by p as in Section 3 the total (portfolio-wide) probability of default, it follows from (3.3a) that the unconditional distribution function $F(s)$ of the score variable can be written as

$$F(s) \;=\; \mathrm{P}[S \le s] = (1-p)\,F_N(s) + p\,F_D(s). \tag{5.1}$$

The equation of the *CAP function* is then given by

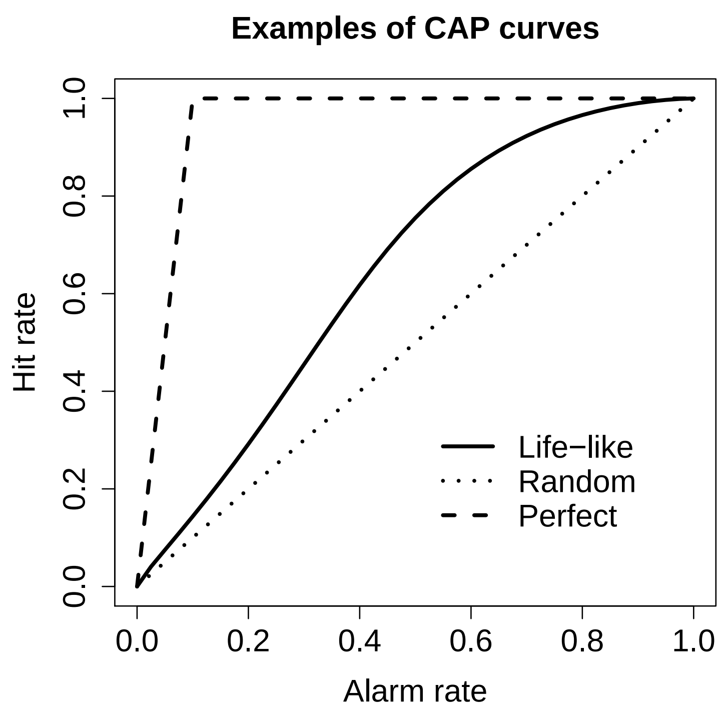
$$CAP(u) \;=\; F_D\big(F^{-1}(u)\big), \quad u \in (0,1). \tag{5.2}$$

The graph of the l can either be drawn by plotting all the points $(u, CAP(u))$, $u \in (0,1)$ or by plotting all the points $(F(s), F_D(s))$, $s \in \mathbb{R}$. The latter parametrization of the CAP curve can still be used when the score distribution function F is not invertible. Figure 3 shows examples of how a CAP curve may look like. The solid curve belongs to the score variable whose two conditional densities are shown in Figure 1. A curve like this could occur in practice. The two other curves in Figure 3 correspond to the so-called *random* (dotted line) and *perfect* (dashed curve) respectively score variables. In case of a random score variable the two conditional densities f_D and f_N are identical. Such a score variable has no discriminatory power at all. In case of a perfect score variable the densities f_D and f_N have disjoint supports, i.e.

$$\{s:\; f_D(s) > 0\} \cap \{s:\; f_N(s) > 0\} \;=\; \emptyset. \tag{5.3}$$

(5.3) implies that the realizable range of the scores of the defaulting borrowers and the realizable range of the scores of the non-defaulting borrowers are disjoint, too. As a consequence, perfect discrimination of defaulters and non-defaulters would be possible.

Figure 3: *Illustrative example of CAP curves of life-like, random and perfect score variables as explained in main text. For the score variable based on the conditional densities shown in Figure 1. Total PD 10 percent.* $AR = 0.336$.



In the context of CAP curves, $F(s)$ is called *alarm rate* associated with the score level s and F_D is called *hit rate* associated with the score level s. These names indicate what happens if all the borrowers with a score equal to or less than some fixed threshold s are considered suspect of default (cut-off rule in the sense of (4.8a) and (4.8b)). The hit rate then reflects which portion of the defaulters will be detected by this procedure. The alarm rate gives the portion of the whole population which will raise suspicion of being prone to default. From these observations it follows that $100\,CAP(u)\%$ indicates the percentage of default-infected borrowers that are found among the first (according to their scores) $100\,u\%$ of all borrowers. A further consequence is that the "perfect" curve in Figure 3 corresponds to a score variable which is of cut-off type in the sense of Section 4.

It seems unlikely that any rating system or score variable from practice will receive a CAP-curve like that from the perfect rating system since this would indicate that it will enable its owners to detect defaulters with certainty. Similarly, it is not very likely to observe in practice a rating system with zero power as a CAP-curve identical to the diagonal would indicate. However, if a rating system is developed for a certain portfolio and then is used on a completely different one, a very low discriminatory power can be the result of such a procedure.

It is easy to show that the CAP function is related to the conditional probability of default given the score via its derivative scaled by the total probability of default.

$$CAP'(u) \;=\; \mathrm{P}[D \,|\, S = F^{-1}(u)] \,/\, p. \tag{5.4}$$

Recall from (3.3b) that by Bayes' formula the conditional probability of default given the score can be represented as a ratio involving the conditional score densities. As a consequence of (5.4) and that representation, the stronger is the growth of $CAP(u)$ for u close to 0 (implying the conditional PD being close to 1 for low scores) and the weaker is the growth of $CAP(u)$ for u close to 1 (implying the conditional PD being close to 0 for high scores), the more differ the conditional densities and the better is the discriminatory power of the underlying score variable.

Accuracy Ratio (AR). From Figure 3, it is intuitively clear that the area between the diagonal line and the CAP curve can be considered a measure of discriminatory power. The random score variable receives area 0, and the life-like score variable obtains an area greater than 0 but less than the area of the perfect score variable. The area between the diagonal and the CAP-curve of the life-like rating system (solid line) can be calculated as the integral from 0 to 1 of the CAP function (5.2) minus 1/2. The area between the curve of the perfect score variable and the diagonal is given by $1/2 - p/2$ when p denotes the total probability of default.

The so-called *Accuracy Ratio* (AR) (also *Gini-coefficient*) is defined as the ratio of the area between the CAP-curve and the diagonal and the area between the perfect CAP curve and the diagonal, i.e.

$$AR \;=\; \frac{2\int_0^1 CAP(u)\,du - 1}{1-p}. \tag{5.5a}$$

Alternatively, the Accuracy Ratio can be described as the difference of two probabilities. Imagine that two borrowers are independently selected at random, one from the defaulters population and the other from the non-defaulters population. The first probability is the probability of the event to observe a higher score for the non-defaulting borrower. The subtracted probability is the probability of the event that the defaulting borrower has the higher score. Then

$$AR \;=\; \mathrm{P}[S_D < S_N] - \mathrm{P}[S_D > S_N], \tag{5.5b}$$

where S_N and S_D are independent and distributed according to F_N and F_D respectively. Obviously, if we assume that in general non-defaulters have the higher scores, we will expect that the first probability is higher than the second as is also indicated by the graph in Figure 3.

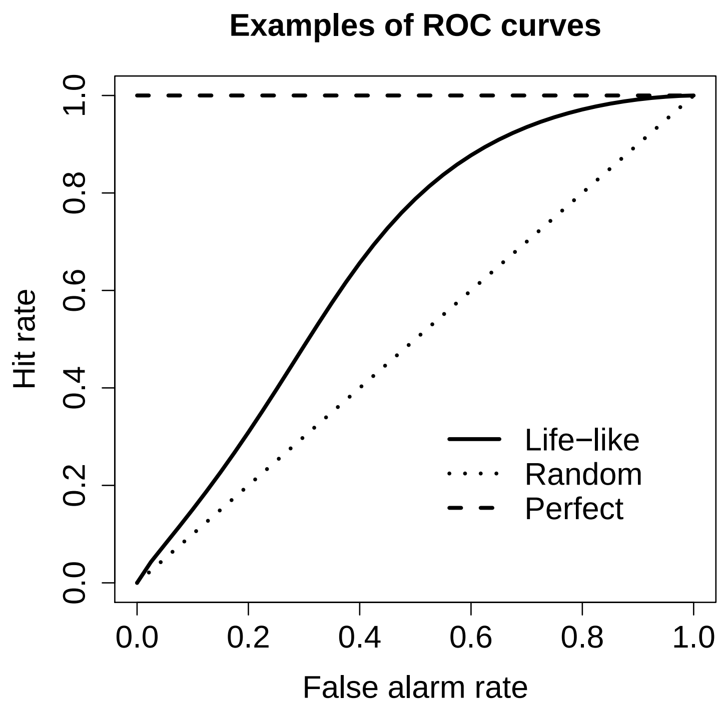
From Figure 3 we can conclude that the discriminatory power of a rating system will be the higher, the larger its Accuracy Ratio is. This follows from (5.4), since a large Accuracy Ratio implies that the PDs for the low scores are large whereas the PDs for the high scores are small.

Receiver Operating Characteristic (ROC). The Receiver Operating Characteristic (or ROC) is another graphical tool for investigating discriminatory power. Define, additionally to the notions of hit rate and alarm rate from the context of the CAP curve, the *false alarm rate* associated with the score level s as the conditional probability $\mathrm{P}[S \le s \,|\, N] = F_N(s)$ that the score of a non-defaulting borrower is less than or equal to this score level. Then the false alarm rate reflects the portion of the non-defaulters population which will be under wrong suspicion when a cut-off rule with threshold s is applied. The equation of the *ROC function* is now given by

$$ROC(u) \;=\; F_D\big(F_N^{-1}(u)\big), \quad u \in (0,1). \tag{5.6a}$$

The graph of the s can either be drawn by plotting all the points $(u, ROC(u))$, $u \in (0, 1)$ or by plotting all the points $(F_N(s), F_D(s))$, $s \in \mathbb{R}$. The latter parametrization of the ROC curve can still be used when the conditional score distribution function F_N is not invertible. In contrast to the case with CAP curves, constructing ROC curves does not involve estimation of the total (or portfolio-wide) probability of default. Figure 4 shows examples of how a ROC curve may look like. The solid curve belongs to the score variable whose two conditional densities are shown in Figure 1. The dotted and the dashed curves correspond to the random score variable and the perfect score variable respectively as in the case of the CAP curves in Figure 3. As a result of

Figure 4: *Illustrative example of ROC curves of life-like, random and perfect score variables as explained in main text. For the life-like score variable based on the conditional densities shown in Figure 1.* $AUC = 0.668$.



this definition, $100\, ROC(u)\%$ indicates the percentage of default-infected borrowers that have been assigned a score which is lower than the highest score of the first (according to their scores) $100\, u\%$ non-defaulters. Alternatively the points on the ROC curve can be characterized as all pairs of type I error probability and power (see Section 4) that can arise when cut-off rules are applied for testing the hypothesis “non-default” against the alternative “default”.

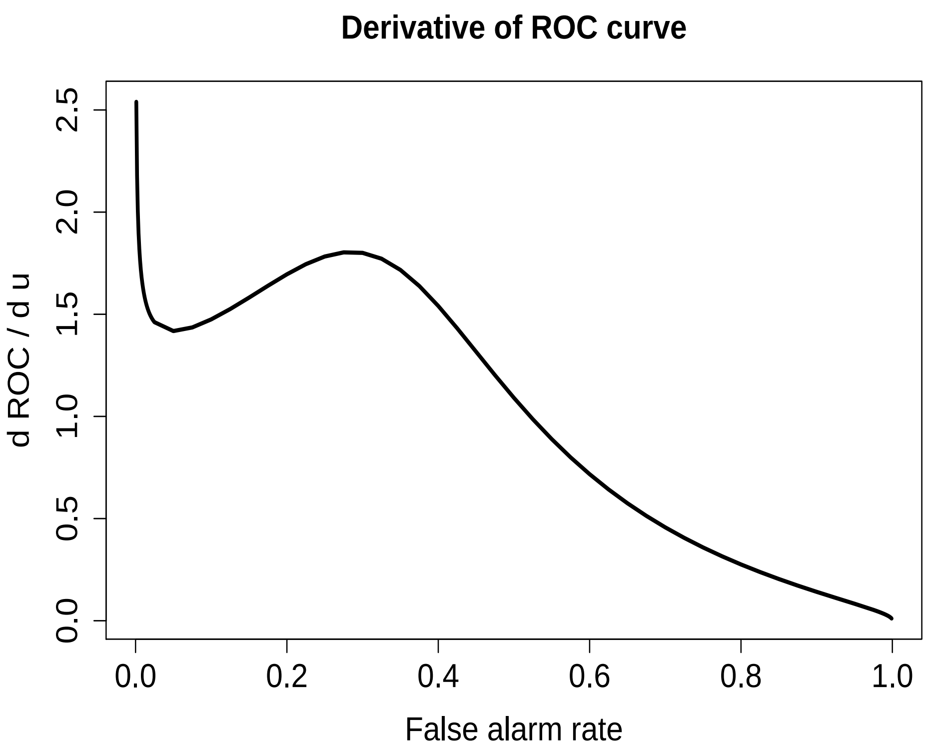
The derivative of the ROC curve turns out to be closely related to the likelihood ratio which was already mentioned in the context of the Neyman-Pearson lemma in Section 4.

$$ROC'(u) \;=\; \frac{f_D(F_N^{-1}(u))}{f_N(F_N^{-1}(u))}, \qquad u \in (0, 1). \tag{5.6b}$$

Hence, the stronger is the growth of $ROC(u)$ for u close to 0 and the weaker is the growth of $ROC(u)$ for u close to 1, the more differ the conditional densities and the better is the discriminatory power of the underlying score variable.

From Section 4 we know that the score variable is of cut-off type and hence optimal in a test-theoretic sense if and only if the likelihood ratio is monotonous. Via (5.6b) this is also equivalent to the CAP curve being concave or convex as concavity and convexity mean that the first derivative is monotonous. If high scores indicate high creditworthiness, the conditional score density f_D is small for high scores and large for low scores and the conditional score density f_N is large for high scores and small for low scores. As a conclusion, the ROC curve of an optimal score variable need to be concave in the case where high scores indicate high creditworthiness. While the lack of concavity of the solid curve in Figure 4 is not very clear, from the graph of its derivative according to (5.6b) in Figure 5 the lack of monotonicity is obvious.

Figure 5: *Derivative of the ROC curve given by the solid line in Figure 4. For the life-like score variable based on the conditional densities shown in Figure 1.*



Area under the curve (AUC). As a further measure of discriminatory power the *Area under the curve* (AUC) is defined as the area between the ROC curve and the axis of abscissa in Figure 4. This area can be calculated as the integral of the ROC curve from 0 to 1, i.e.

$$AUC \;=\; \int_0^1 ROC(u)\,du. \tag{5.7a}$$

Alternatively, the AUC can be described as a probability, namely that the score of a non-defaulter selected at random is higher than the score of an independently selected defaulting borrower. Hence,

$$AUC \;=\; \mathrm{P}[S_D < S_N], \tag{5.7b}$$

where S_N and S_D are independent and distributed according to F_N and F_D respectively. Moreover, it can be proved (cf., for instance Engelmann et al., 2003) that the AUC is just an affine transformation of AR, namely

$$AUC \quad = \quad \frac{AR+1}{2}. \tag{5.7c}$$

As a consequence of this last observation, the higher AUC the higher is the discriminatory power of the rating system under consideration, as is the case for AR. Moreover, maximizing AUC is equivalent to maximizing discriminatory power as maximizing the area under the ROC curve due (5.6b) and (3.3b) results in high PDs for small scores and low PDs for large scores. Additionally, (5.7c) shows that the value of AR – like the value of AUC – depends on the conditional densities of the score variable given the state of the borrower but not on the total probability of default in the portfolio.

There is also an important consequence from the representation of AUC as a probability. The non-parametric Mann-Whitney test (see, e.g. Sheskin, 1997) for the hypothesis that one distribution is stochastically greater than another can be applied as a test on whether there is discriminatory power at all or not. Additionally, a Mann-Whitney-like test for comparing the discriminatory power values of two or more rating systems is available (cf. Engelmann et al., 2003).

Error rates as measures of discriminatory power. We have seen that the ROC curve may be interpreted as a "type I error level"-power diagram related to cut-off decision rules in the sense of (4.8a) and (4.8b), based on the score variable under consideration. Another approach to measuring discriminative power is to consider only total probabilities of error instead of type I and II error probabilities separately.

The first example of an error-rate based measure of discriminatory power is the *Baysian error rate*. It is defined as the minimum total probability of error that can be reached when cut-off rules are applied.

$$\begin{aligned}
\text{Baysian error rate} &= \min_s \mathrm{P}[\text{Erroneous decision when cut-off rule with threshold } s \text{ is applied}] \\
&= \min_s \big(\mathrm{P}[Z=D]\,\mathrm{P}[S>s\,|\,Z=D] + \mathrm{P}[Z=N]\,\mathrm{P}[S\le s\,|\,Z=N]\big) \\
&= \min_s \big(p\,\big(1-F_D(s)\big) + (1-p)\,F_N(s)\big)\,. \qquad (5.8a)
\end{aligned}$$

In the special case of a hypothetical total PD of 50 percent the Baysian error rate is called *classification error*. Assume that defaulters tend to receive smaller scores than non-defaulters, or, technically speaking, that F_D is stochastically smaller than F_N (i.e. $F_D(s) \ge F_N(s)$ for all s). The classification error can then be written as

$$\text{Classification error} \;=\; 1/2 - 1/2\,\max_s |F_D(s) - F_N(s)|. \tag{5.8b}$$

The maximum term on the right-hand side of (5.8b) is just the population version of the well-known *Kolmogorov-Smirnov* statistic for testing whether the two distributions F_D and F_N are identical. The conditional distributions of the score variable being identical means that the score variable has not any discriminatory power. Thus, the classification error is another example of a measure of discriminatory power for which well-known and efficient test procedures are available. The so-called *Pietra-index* reflects the maximum distance of a ROC curve and the diagonal. In the case where the likelihood ratio f_D/f_N is a monotonous function, the Pietra-index can be written as an affine transformation of the Kolmogorov-Smirnov statistic and therefore is equivalent to it in a statistical sense.

If the likelihood ratio is monotonous, the Kolmogorov-Smirnov statistic has an alternative representation as follows:

$$\max_s |F_D(s) - F_N(s)| \,=\, 1/2 \int_{-\infty}^{\infty} |f_D(s) - f_N(s)|\, ds \,\in\, [0, 1/2]. \tag{5.8c}$$

This representation is interesting because it allows to compare the Kolmogorov-Smirnov statistic with the *information value*, a discrepancy measure which is based on relative entropies. We will not explain here in detail the meaning of relative entropy. What is important here, is the fact that the information value can be written in a way that suggests to interpret the information value as something like a "weighted Kolmogorov-Smirnov" statistic.

$$\begin{aligned}\text{Information value} &= \mathrm{E}\big[\log \frac{f_D(S)}{f_N(S)} \,|\, D\big] + \mathrm{E}\big[\log \frac{f_N(S)}{f_D(S)} \,|\, N\big] \\ &= \int_{-\infty}^{\infty} (f_D(s) - f_N(s))(\log f_D(s) - \log f_N(s))\, ds \\ &\in\, [0, \infty).\end{aligned} \tag{5.8d}$$

Note that the information value is also called *divergence* or *stability index*. Under the notion stability index it is sometimes used as a tool to monitor the stability of score variables over time.

Measuring discriminatory power as variation of the PD conditional on the score. So far we have considered measures of discriminatory power which are intended to express the discrepancy of the conditional distributions of the scores for the defaulters population and the non-defaulters population respectively. Another philosophy of measuring discriminatory power is based on measuring the variation of the conditional PD given the scores. Let us first consider the two extreme cases.

A score variable has no discriminatory power at all if the two conditional densities of the score distribution (as illustrated in Figure 1) are identical. In that case the borrowers' score variable S and state variable Z are stochastically independent. As a consequence, the conditional PD given the score is constant and equals the total PD.

$$\mathrm{P}[D \,|\, S] \,=\, p. \tag{5.9a}$$

One could also say that the score variable S does not bear any information about potential default. Obviously, such a score variable would be considered worthless.

The other extreme case is the case where the conditional PD given the scores takes on the values 0 and 1 only.

$$\mathrm{P}[D \,|\, S] \,=\, \mathbf{1}_D \,=\, \begin{cases} 1, & \text{if borrower defaults;} \\ 0, & \text{if borrower remains solvent.} \end{cases} \tag{5.9b}$$

This would be an indication of a perfect score variable as in such a case there were no uncertainty about the borrowers' future state any more. In practice, none of these two extreme cases will occur. The conditional PD given the score will in general neither take on the values 0 and 1 nor will it be constant either.

In regression analysis, the determination coefficient R^2 measures the extent to which a set of explanatory variables can explain the variance of the variable which is to be predicted. A score variable or the grades of a rating system may be considered explanatory variables for the default state indicator. The conditional PD given the score is then the best predictor of the default indicator by the score in the sense of (3.5). Its variance can be compared to the variance of the default indicator in order to obtain an R^2 for this special situation.

$$R^2 = \frac{\mathrm{var}[\mathrm{P}[D\,|\,S]]}{\mathrm{var}[\mathbf{1}_D]} = \frac{\mathrm{var}[\mathrm{P}[D\,|\,S]]}{p\,(1-p)} = 1 - \frac{\mathrm{E}\big[(\mathbf{1}_D - \mathrm{P}[D\,|\,S])^2\big]}{p\,(1-p)} \in [0,1]. \tag{5.10a}$$

The closer the value of R^2 is to one, the better the score S can explain the variation of the default indicator. In other words, if R^2 is close to one, a high difference in the score values does more likely indicate a corresponding difference in the values of the default indicator variable. Obviously, maximizing R^2 is equivalent to maximizing $\mathrm{var}[\mathrm{P}[D\,|\,S]]$ and to minimizing $\mathrm{E}\big[(\mathbf{1}_D - \mathrm{P}[D\,|\,S])^2\big]$.

The sum over all borrowers of the squared differences of the default indicators and the conditional PDs given the scores divided by the sample size is called *Brier score.*

$$\text{Brier score} \;=\; \frac{1}{n}\sum_{i=1}^{n}\big(\mathbf{1}_{D_i} - \mathrm{P}[D\,|\,S = S_i]\big)^2. \tag{5.10b}$$

The Brier score is a natural estimator of $\mathrm{E}\big[(\mathbf{1}_D - \mathrm{P}[D\,|\,S])^2\big]$ which is needed for calculating the R^2 of the score variable under consideration. Note that as long as default or non-default of borrowers cannot be predicted with certainty (i.e. as long as (5.9b) is not satisfied) $\mathrm{E}\big[(\mathbf{1}_D - \mathrm{P}[D\,|\,S])^2\big]$ will not equal 0.

In practice, the development of a rating system or score variable involves both an optimization procedure (such as maximizing R^2) and an estimation exercise (estimating the PDs given the scores $\mathrm{P}[D\,|\,S = s]$). The Brier score can be used for both purposes. On the one hand selecting an optimal score variable may be conducted by minimizing $\mathrm{E}\big[(\mathbf{1}_D - \mathrm{P}[D\,|\,S])^2\big]$, which usually also involves estimating $\mathrm{P}[D\,|\,S = s]$ for all realizable score values. On the other hand, when the score variable S has already been selected, the Brier score may be used for calibration purposes (see Section 6).

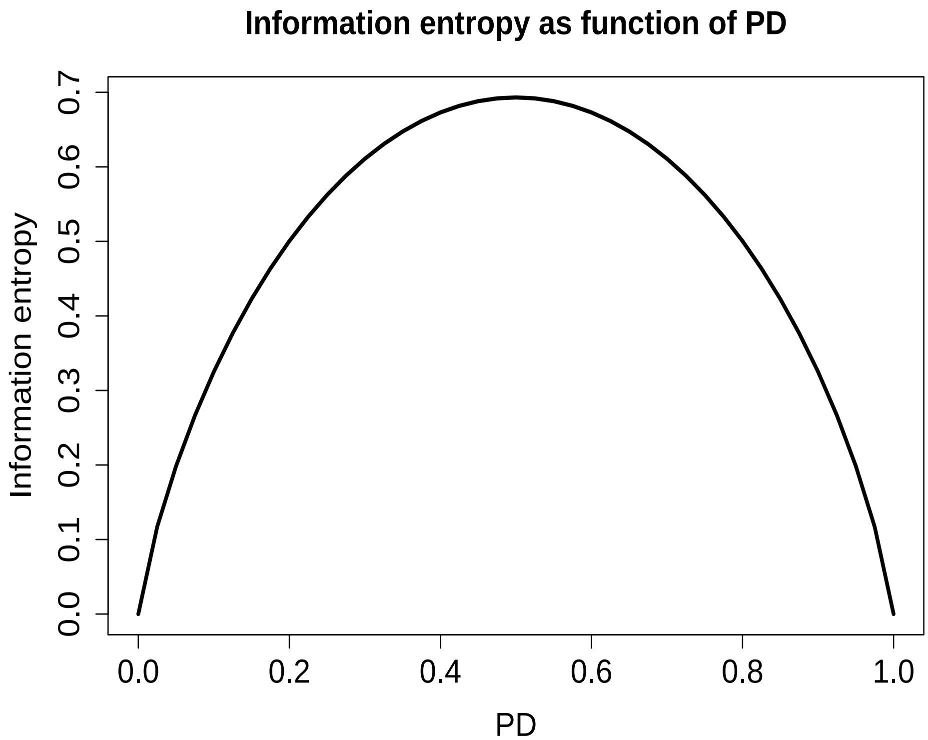
Further entropy measures of discriminatory power. Besides the information value defined in (5.8d) sometimes also other entropy-based measures of discriminatory power are used in practice.

For any event with probability p its *information entropy* is defined as

$$H(p) \;=\; -\big(p\,\log p + (1-p)\,\log(1-p)\big) \tag{5.11a}$$

Note from Figure 6 that $H(p)$ is close to 0 if and only if p is close to 0 or close to 1. As a consequence, information entropy can be regarded as a measure of uncertainty of the underlying event. When discriminatory power of a score variable has to be measured, it can be useful to consider the information entropy applied to the conditional PD given the scores, i.e. $H\big(\mathrm{P}[D\,|\,S]\big)$. If then the average value of the information entropy is close to zero, the conditional PD given the scores will be close to zero or to one in average, indicating high discriminatory power. Formally, the average information entropy of the conditional PD is described as *conditional entropy* H_S

Figure 6: *Graph of function* $p \mapsto -\big(p \log p + (1-p) \log(1-p)\big)$.



which is defined as the expectation of the information entropy applied to the conditional PD given the scores.

$$H_S \;=\; \mathrm{E}\big[H\big(\mathrm{P}[D \,|\, S]\big)\big]. \tag{5.11b}$$

As both the conditional PD given the scores as well as the calculation of the expectation depend on the portfolio-wide total PD, it is not sensible to compare directly the conditional entropy values of score variables from populations with different portions of defaulters. However, it can be shown by Jensen's inequality that the conditional entropy never exceeds the information entropy of the total probability of default of the population under consideration. Therefore, by using the *conditional information entropy ratio* (CIER), defined as ratio of information entropy of the total PD minus conditional entropy of the conditional PDs and the information entropy of the total PD, conditional entropy values of different score variables can be made commensurable.

$$\mathrm{CIER} \;=\; \frac{H(p) - H_S}{H(p)} \;\in\; [0,1] \tag{5.11c}$$

The closer the value of CIER is to one, the more information about default the score variable S bears, in the sense of providing conditional PDs given the scores which are close to 0 or to 1.

6 Calibration of rating systems

The issue with calibration of rating systems or score variables is how accurate the estimates of the conditional default probability given the score are. Supervisors, in particular, require that the estimates are not too low when they are used for determining regulatory capital requirements. In the following, we will consider some tests on calibration that are conditional on the state of the economy. These are the binomial test, the Hosmer-Lemeshow test and the Spiegelhalter test. As an example for unconditional tests, we will then discuss a normal approximate test.

Conditional versus unconditional tests. The notions of conditional and unconditional tests in the context of validation for Basel II can be best introduced by relating these notions to the notions of PIT and TTC PD estimates.

PD estimates can be based (or, technically speaking, conditioned) on the current state of the economy, for instance by inclusion of macro-economic co-variates in a regression process. The co-variates are typically the growth rate of the gross domestic product, the unemployment rate or similar indices. The resulting PD estimates are then called *Point-In-Time* (PIT). With such estimates, given an actual realization of the co-variates, an assumption of independence of credit events may be adequate, because most of their dependence might have been captured by incorporating the economic state variables in the PDs estimates.

In contrast, unconditional PD estimates are not based on a current state of the economy. Unconditional PDs that are estimated based on data from a complete economic cycle are called *Through-The-Cycle* (TTC). When using unconditional PDs, no assumption of independence can be made, since then the variation of the observed default rates cannot be any longer explained by the variation of conditional PDs which are themselves random variables.

Binomial test. Consider one fixed rating grade specified by a range $s_0 \leq S \leq s_1$, as described, for instance, in (3.7a) and (3.7b). It is then reasonable to assume that an average PD q has been forecast for the rating grade under consideration. Let n be the number of borrowers that have been assigned this grade.

If the score variable is able to reflect to some extent the current state of the economy, default events among the borrowers may be considered stochastically independent. Under such an independence assumption, the number of defaults in the rating grade is binomially distributed with parameters n and q. Hence the *binomial test* (cf., e.g. Brown et al., 2001) may be applied to test the hypothesis "the true PD of this grade is not greater than the forecast q". If the number of borrowers within the grade and the hypothetical PD q are not too small, thanks to the central limit theorem under the hypothesis the binomial distribution can be approximated with a normal distribution. As already mentioned, for this approximation to make sense is important that the independence assumption is justified. This will certainly not be the case when the PDs are estimated through-the-cycle. The following example illustrates what then may happen.

Example. Assume that 1000 borrowers have been assigned the rating grade under consideration. The bank forecasts for this grade a PD of 1 percent. One year after the forecast 19 defaults are observed.

If we assume independence of the default events, with a PD of 1 percent the probability to observe 19 or more defaults is 0.7 percent. Hence, the hypothesis that the true PD is not greater than 1 percent can be rejected with 99 percent confidence. As a consequence, we would conclude that the bank's forecast was too optimistic.

Assume now that the default events are not independent. For the purpose of illustration, the dependence then can be modelled by means of a normal copula with uniform correlation 5 percent (see, e.g. Pluto and Tasche, 2005, for details of the one-factor model). Then, with a PD of 1 percent, the probability to observe 19 or more defaults is 11.1 percent. Thus, the hypothesis that the true PD is not greater than 1 percent cannot be rejected with 99 percent confidence. As a consequence, we would accept the bank's forecast as adequate.

Hosmer-Lemeshow test. The binomial test can be appropriate to check a single PD forecast. However, if – say – twenty PDs of rating grades are tested stand-alone, it is quite likely that at least one of the forecasts will be erroneously rejected. In order to have at least control over the probability of such erroneous rejections, joint tests for several grades have to be used.

So, assume that there are PD forecasts $q_1, \ldots, q_k$ for rating grades $1, \ldots, k$. Let n_i denote the number of borrowers with grade i and d_i denote the number of defaulted borrowers with grade i. The *Hosmer-Lemeshow statistic* H for such a sample is the sum of the squared differences of forecast and observed numbers of default, weighted by the inverses of the theoretical variances of the default numbers.

$$H = \sum_{i=1}^{k} \frac{(n_i\, q_i - d_i)^2}{n_i\, q_i\,(1 - q_i)}. \tag{6.1}$$

Under the usual assumptions on the appropriateness of normal approximation (like independence, enough large sample size), the Hosmer-Lemeshow statistic is χ^2_k-distributed under the hypothesis that all the PD forecasts match the true PDs. This fact can be used to determine the critical values for testing the hypothesis of having matched the true PDs. However, also for the Hosmer-Lemeshow test, the assumption of independence is crucial. Additionally, there may be an issue of bad approximation for rating grades with small numbers of borrowers.

Spiegelhalter test. If the PDs of the borrowers are individually estimated, both the binomial test and the Hosmer-Lemeshow test require averaging the PDs of borrowers that have been assigned the same rating grade. This procedure can entail some bias in the calculation of the theoretical variance of the number of defaults. With the Spiegelhalter test, one avoids this problem. As for the binomial and Hosmer-Lemeshow test, also for the Spiegelhalter test independence of the default events is assumed. As mentioned earlier, if the PD are estimated point in time, the independence assumption may be justified.

We consider borrowers $1, \ldots, n$ with scores s_i and PD estimates p_i. Given the scores, the borrowers are considered to default or remain solvent independently. Recall the notion of Brier score from (5.10b). In contrast to the situation when a rating system or score variable is developed, for the purpose of validation we assume that realizations of the ratings are given and hence non-random. Therefore we can drop the conditioning on the score realizations in the notation. In the context of validation, the Brier score is also called *Mean squared error (MSE).*

$$MSE = 1/n \sum_{i=1}^{n} (\mathbf{1}_{D_i} - p_i)^2, \tag{6.2a}$$

where $\mathbf{1}_{D_i}$ denotes the default indicator as in (5.9b). The null hypothesis for the test is "all PD forecasts match exactly the true conditional PDs given the scores", i.e. $p_i = \mathrm{P}[D_i \,|\, S_i = s_i]$ for all i.

It can be shown that under the null we have

$$\mathrm{E}[MSE] = 1/n \sum_{i=1}^{n} p_i\,(1-p_i) \qquad \text{and} \tag{6.2b}$$

$$\mathrm{var}[MSE] = n^{-2} \sum_{i=1}^{n} p_i\,(1-p_i)\,(1-2\,p_i)^2. \tag{6.2c}$$

Under the assumption of independence given the score values, according to the central limit theorem, the distribution of the standardized mean squared error

$$Z \;=\; \frac{MSE - \mathrm{E}[MSE]}{\sqrt{\mathrm{var}[MSE]}} \tag{6.2d}$$

is approximately standard normally distributed under the null. Thus, a joint test of the hypothesis "the calibration of the PDs with respect to the score variable is correct" can be conducted (see Rauhmeier and Scheule, 2005, for examples from practice).

Testing unconditional PDs. As seen before by example, for unconditional PD estimates assuming independence of the defaults for testing the adequacy of the estimates could result in too conservative tests. However, if a time-series of default rates is available, assuming independence over time might be justifiable. Taking into account that unconditional PD estimates usually are constant[5] over time, a simple test can be constructed that does not involve any assumption of cross-sectional independence among the borrowers within a year. We consider a fixed rating grade with n_t borrowers (thereof d_t defaulters) in year $t = 1, \ldots, T$. Additionally, we assume that the estimate q of the PD common to the borrowers in the grade is of TTC type and constant over time, and that defaults in different years are independent. In particular, then the annual default rates d_t/n_t are realization of independent random variables. The standard deviation σ of the default rates can in this case be estimated with the usual unbiased estimator

$$\hat{\sigma}^2 \;=\; \frac{1}{T-1} \sum_{t=1}^{T} \left(\frac{d_t}{n_t} - \frac{1}{T} \sum_{\tau=1}^{T} \frac{d_\tau}{n_\tau} \right)^2 . \tag{6.3a}$$

If the number T of observations is not too small, and under the hypothesis that the true PD is not greater than q, the standardized average default rate is approximately standard normally distributed. As a consequence, the hypothesis should be rejected if the average default rate is greater than q plus a critical value derived by this approximation. Formally, reject "true PD $\leq\ q$" at level α if

$$\frac{1}{T} \sum_{\tau=1}^{T} \frac{d_\tau}{n_\tau} \;>\; q + \frac{\hat{\sigma}}{\sqrt{T}}\, \Phi^{-1}(1-\alpha). \tag{6.3b}$$

As mentioned before, the main advantage of the *normal test* proposed here is that no assumption on cross-sectional independence is needed. Moreover, the test procedure seems even to be robust against violations of the assumption of inter-temporal independence, in the sense that the test results still appear reasonable when there is weak dependence over time. More critical appears the assumption that the number T of observations is large. In practice, time series of length five to ten years do not seem to be uncommon. In Tables 1 and 2 we present the results of

[5] Blochwitz et al. (2004) provide a modification of the test for the case of non-constant PD estimates.

an illustrative Monte-Carlo simulation exercise in order to give an impression of the impact of having a rather short time series.

Table 1
Estimated PD = 2%, $T = 5$, $\alpha = 1\%$

True PD	Rejection rate
1.0%	0.00%
1.5%	0.01%
2.0%	2.05%
2.5%	19.6%
5.0%	99.2%

Table 2
Estimated PD = 2%, $T = 5$, $\alpha = 10\%$

True PD	Rejection rate
1.0%	0.00%
1.5%	0.60%
2.0%	7.96%
2.5%	30.1%
5.0%	99.2%

The exercise whose results are reflected in Tables 1 and 2 was conducted in order to check the quality of the normal approximation for the test of the unconditional PDs according to (6.3b). For two different type I error probabilities the tables present the true rejection rates of the hypothesis "true PD not greater than 2 percent" for different values of the true PDs. By construction of the test, the rejection rates ought to be not greater than the given error probabilities as long as the true PDs are not greater than 2 percent. For the smaller error probability of 1 percent this seems to be a problem, but not a serious one. However, the tables also reveal that the power of the test is rather moderate. Even if the true PD is so clearly greater than the forecast PD as in the case of 2.5 percent, the rejection rates are 19.6 and 30.1 percent respectively only.

7 Conclusions

With regard to measuring discriminatory power, the Accuracy Ratio and the Area under the Curve seem promising[6] tools since their statistical properties are well investigated and they are available together with many auxiliary features in most of the more popular statistical software packages.

With regard to testing calibration, for conditional PD estimates powerful tests like the binomial, the Hosmer-Lemeshow and the Spiegelhalter test are available. However, their appropriateness strongly depends on an independence assumption which needs to be justified on a case by case basis. Such independence assumptions can at least partly be avoided, but at the price of losing power as illustrated with a test procedure based on a normal approximation.

References

Basel Committee on Banking Supervision (BCBS) (2004) Basel II: International Convergence of Capital Measurement and Capital Standards: a Revised Framework.
`http://www.bis.org/publ/bcbs107.htm`

[6]The selection of the topics and the point of view taken in this chapter is primarily a regulatory one. This is caused by the author's background in a regulatory authority. However, the presentation does not reflect any official regulatory thinking. The regulatory bias should be kept in mind when the following conclusions are read. A procedure which may be valuable for regulatory purposes need not necessarily also be appropriate for bank-internal applications.

Basel Committee on Banking Supervision (BCBS) (2005a) Update on work of the Accord Implementation Group related to validation under the Basel II Framework. http://www.bis.org/publ/bcbs_nl4.htm

Basel Committee on Banking Supervision (BCBS) (2005b) Studies on the Validation of Internal Rating Systems (revised). Working Paper No. 14. http://www.bis.org/publ/bcbs_wp14.htm

Blochwitz, S., Hohl, S., Wehn, C. and D. Tasche (2004) Validating Default Probabilities on Short Time Series. *Capital & Market Risk Insights, Federal Reserve Bank of Chicago.*

Brown, L., Cai, T. and A. Dasgupta (2001) Interval Estimation for a Binomial Proportion. *Statistical Science* **16**(2), 101-133.

Casella, G. and R. L. Berger (2001) *Statistical inference.* Second edition. Duxberry, Pacific Grove.

Committee of European Banking Supervisors (CEBS) (2005) Guidelines on the implementation, validation and assessment of Advanced Measurement (AMA) and Internal Ratings Based (IRB) Approaches. http://www.c-ebs.org/pdfs/CP10rev.pdf

Engelmann, B., Hayden, E. and D. Tasche (2003) Testing rating accuracy. *Risk* **16**(1), 82-86.

Pluto, K. and D. Tasche (2005) Thinking positively. *Risk* **18**(8), 72-78.

Rauhmeier, R. and H. Scheule (2005) Rating properties and their implications for Basel II capital. *Risk* **18**(3), 78-81.

Sheskin, D. J. (1997) *Handbook of parametric and nonparametric statistical procedures.* CRC Press, Boca Raton.

Tasche, D. (2002) Remarks on the monotonicity of default probabilities. *Working paper.* http://arxiv.org/abs/cond-mat/0207555