

Measuring risk concentration

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- Capital allocation (Slide 4)
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Introduction (1)

- The CRD* on risk concentrations:

The **concentration risk** arising from exposures to counterparties, groups of connected counterparties, and counterparties in the same economic sector, geographic region or from the same activity or commodity, the application of credit risk mitigation techniques, and including in particular risks associated with large indirect credit exposures (e.g. to a single collateral issuer), **shall be addressed and controlled** by means of written policies and procedures.

- **Economic capital** is accepted as measure of concentrations[†]:

Institutions should assess the amount of internal capital which they consider to be adequate to hold against the level of concentration risk in their portfolio.

*Capital Requirements Directive 2006/48/EC Annex V para. 7
http://ec.europa.eu/internal_market/bank/regcapital/index_en.htm

[†]Committee of European Banking Supervisors, “Additional Technical Guideline on Concentration Risk” Principle 5, <http://www.c-eps.org/standards.htm>

Introduction (2)

- In larger institutions, usually economic capital is the output of a stochastic credit portfolio risk model:

$$L = \sum_{i=1}^n L_i. \quad (2.1)$$

- $L_i \geq 0$ loss with loan i , L portfolio-wide loss.
- L, L_1, \dots, L_n random variables.
- **Value-at-Risk** (VaR) at level α (e.g. $\alpha = 99.9\%$):

$$\text{VaR}_\alpha(L) = \min\{\ell : P[L \leq \ell] \geq \alpha\} \quad (2.2)$$

- $\text{VaR}_\alpha(L)$ is interpreted as capital buffer (\approx economic capital) needed to guarantee bank's solvency with probability α .

Introduction (3)

- To identify concentrations, economic capital (VaR) has to be decomposed into risk-sensitive components.
- **Capital allocation:** Risk-sensitive decomposition

$$\text{VaR}_\alpha(L) = \sum_{i=1}^n \text{VaR}_\alpha(L_i | L) \quad (3.1)$$

- In general, there are no closed-form representations of $\text{VaR}_\alpha(L)$ and **risk contributions** $\text{VaR}_\alpha(L_i | L)$.
- Instead: Generate sample $(L^{(t)}, L_1^{(t)}, \dots, L_n^{(t)})$, $t = 1, \dots, T$ by Monte-Carlo simulation.
- Estimators for $\text{VaR}_\alpha(L)$ and $\text{VaR}_\alpha(L_i | L)$?

Capital allocation (1)

- Risk sensitivity \Rightarrow^*

$$\text{VaR}_\alpha(L_i | L) = \left. \frac{\partial \text{VaR}_\alpha(L + h L_i)}{\partial h} \right|_{h=0} \quad (4.1)$$

- Joint distribution of (L, L_i) “smooth” \Rightarrow

$$\left. \frac{\partial \text{VaR}_\alpha(L + h L_i)}{\partial h} \right|_{h=0} = \mathbb{E}[L_i | L = \text{VaR}_\alpha(L)] \quad (4.2)$$

- $\mathcal{L}(L | L > 0)$ continuous \Rightarrow

$$\mathbb{E}[L_i | L = \text{VaR}_\alpha(L)] \neq \frac{\mathbb{E}[L_i \mathbf{1}_{\{L = \text{VaR}_\alpha(L)\}}]}{\mathbb{P}[L = \text{VaR}_\alpha(L)]}$$

*Tasche, D. (1999) Risk contributions and performance measurement. Working paper, Technische Universität München.

Capital allocation (2)

- Gouriéroux et al. (2000)* apply kernel estimation methods for VaR contributions when optimizing returns in a portfolio of **stocks**.
- Glasserman (2005)† applies importance sampling for estimating VaR contributions in a **credit** portfolio with **discrete** loss distribution.
- **New approach**‡: For credit portfolio with **continuous** loss distribution, combine kernel estimation with importance sampling.

*Gouriéroux, C., Laurent, J. P. and O. Scaillet (2000) Sensitivity analysis of Values at Risk. *Journal of Empirical Finance*, **7**, 225-245.

†Glasserman, P. (2005) Measuring Marginal Risk Contributions in Credit Portfolios. *Journal of Computational Finance* **9**, 1-41.

‡D. Tasche, (2006) Capital allocation for credit portfolios with kernel estimators. *Working paper*.

Kernel estimators (1)

- Assume that x_1, \dots, x_T is sample of realisations of random variable X with density f .
- Kernel **(Rosenblatt-Parzen)** estimator* \hat{f}_h with bandwidth $h > 0$ for f :
 - X^* with $P[X^* = x_t] = 1/T$, ξ with kernel (density) φ , independent
 - \hat{f}_h density of $X^* + h\xi$:

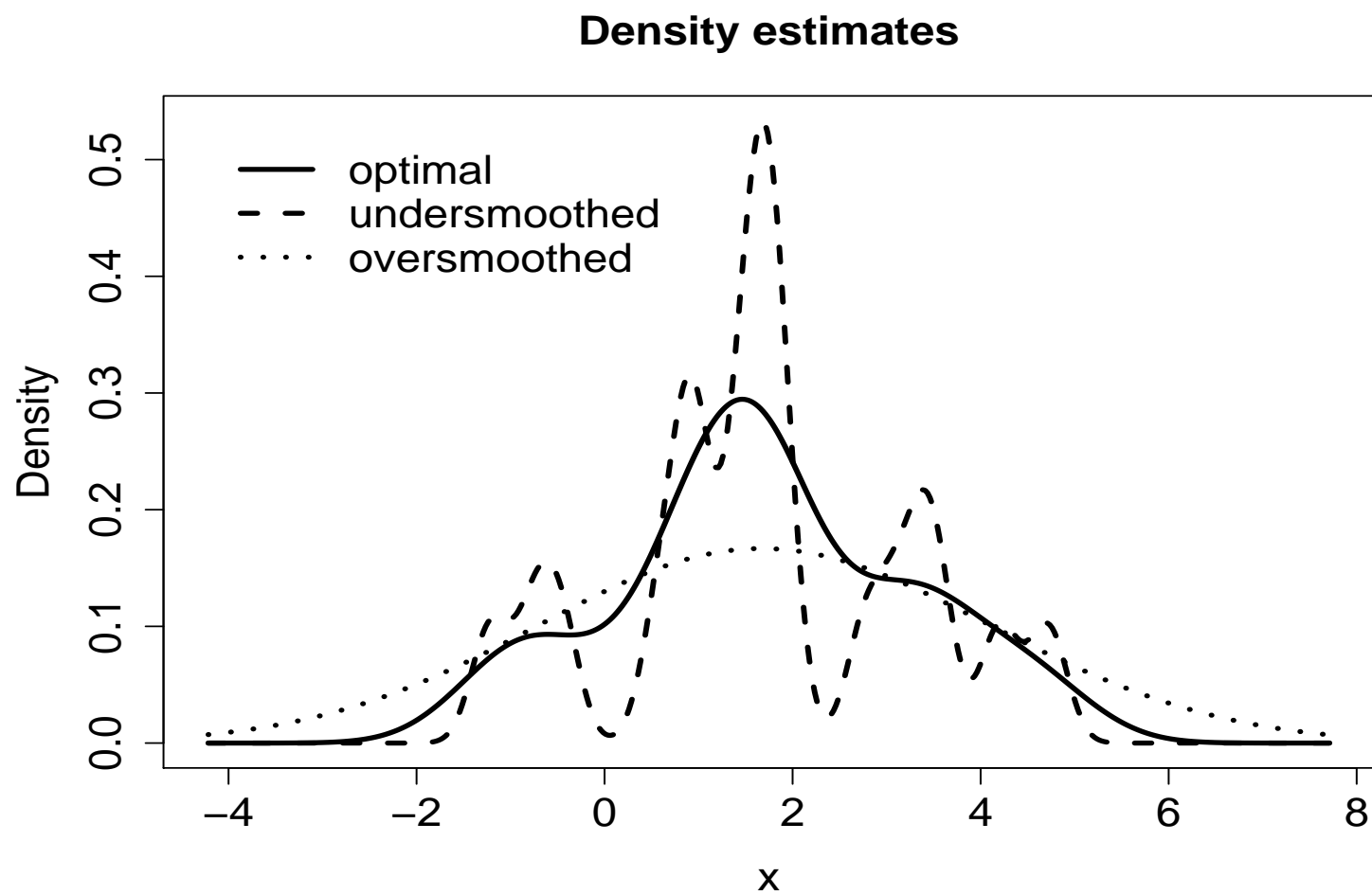
$$\hat{f}_h(x) = \frac{1}{hT} \sum_{t=1}^T \varphi\left(\frac{x-x_t}{h}\right) \quad (6.1)$$

- f and φ smooth, $h_T \rightarrow 0$, $h_T T \rightarrow \infty \Rightarrow$

$$\lim_{T \rightarrow \infty} E_{(X_1, \dots, X_T)} [(f(x) - \hat{f}_{h_T}(x))^2] = 0, \quad x \in \mathbb{R}$$

*Cf., e.g., Pagan, A. and A. Ullah (1999) *Nonparametric econometrics*. Cambridge University Press: Cambridge.

Kernel estimators (2)



Kernel estimators (3)

- Assume that $(x_1, y_1), \dots, (x_T, y_T)$ is sample of realisations of random vector (X, Y) where X has a density f .
- Kernel (**Nadaraya-Watson**) estimator $\hat{E}_h[Y | X = x]$ with bandwidth $h > 0$ for $E[Y | X = x]$:
 - (X^*, Y^*) with $P[(X^*, Y^*) = (x_t, y_t)] = 1/T$,
 ξ with kernel (density) φ , independent
 - $\hat{E}_h[Y | X = x]$ expectation of Y^* conditional on $X^* + h\xi = x$:

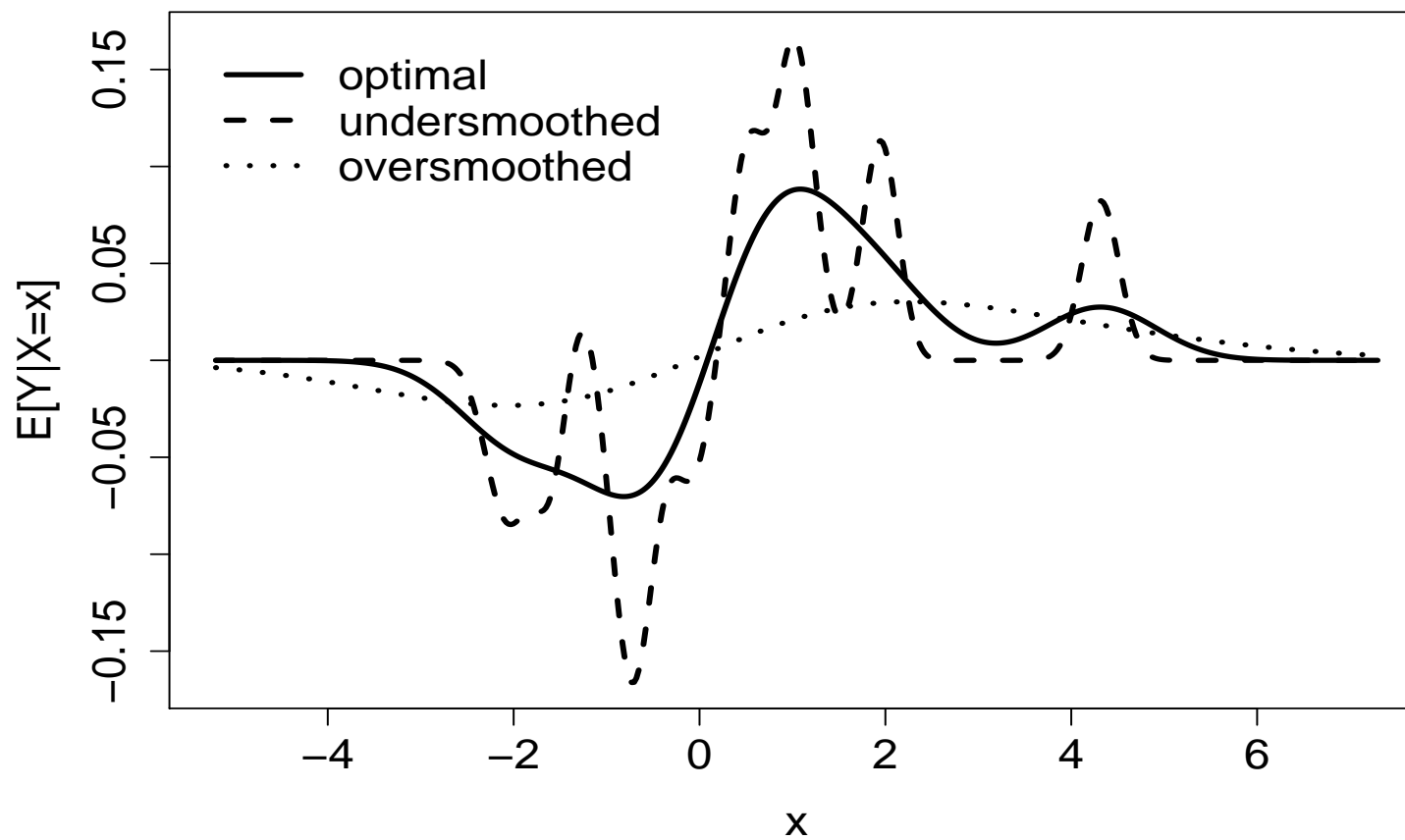
$$\hat{E}_h[Y | X = x] = \frac{\sum_{t=1}^T y_t \varphi\left(\frac{x-x_t}{h}\right)}{\sum_{t=1}^T \varphi\left(\frac{x-x_t}{h}\right)} \quad (8.1)$$

- Distribution of (X, Y) and φ smooth, $h_T \rightarrow 0$, $h_T T \rightarrow \infty \Rightarrow$

$$\lim_{T \rightarrow \infty} E_{(X_1, Y_1), \dots, (X_T, Y_T)} \left[\left(E[Y | X = x] - \hat{E}_h[Y | X = x] \right)^2 \right] = 0, \quad x \in \mathbb{R}$$

Kernel estimators (4)

Conditional expectation estimates



Importance sampling for credit risk (1)

- Basic idea: Shift loss distribution $\ell \mapsto P[L \leq \ell]$ such that event $\{L \geq \text{VaR}_\alpha(L)\}$ becomes much more likely.

- **Exponential tilting:** Replace P by Q with

$$Q[L \in A] = \frac{E_P[\exp(tL) \mathbf{1}_{\{L \in A\}}]}{E_P[\exp(tL)]},$$

with “suitable” $t \in \mathbb{R}$.

- Case of independent L_i with default probabilities $p_i = P[L_i > 0]$, exposure sizes s_i , and “mean LGD” $E[LG D_i]$: Replace p_i by

$$q_i = \frac{\exp(t s_i E[LG D_i]) p_i}{\exp(t s_i E[LG D_i]) p_i + 1 - p_i}.$$

Importance sampling for credit risk (2)

- Assumption: P has μ -density $f > 0$, Q has μ -density $g > 0$.

- Likelihood ratio: $R = \frac{f}{g}$.

- Lemma **(conditional expectation under importance sampling)**:

$$E_P[L_i | L = \ell] = \frac{E_Q[R L_i | L = \ell]}{E_Q[R | L = \ell]}. \quad (11.1)$$

- Model (8.1) \Rightarrow use same bandwidth h from (6.1) for all estimates $\hat{E}_Q[R L_i | L = \ell]$ and $\hat{E}_Q[R | L = \ell]$.

Example: credit risk with continuous losses (1)

- Model (2.1), distribution approximation by Monte-Carlo simulation.
 - Independent L_i with default probabilities $P[L_i > 0] = p_i$.
 - $L_i = s_i LGD_i \mathbf{1}_{D_i}$ with deterministic exposure size s_i and continuous independent loss given default $LGD_i \in (0, 1)$.
- Consequences:
 - Portfolio-wide loss L with $P[L = 0] > 0$ and continuous distribution conditional on $\{L > 0\} = \{L_1 > 0\} \cup \dots \cup \{L_n > 0\}$.
 - Kernel estimators for $\text{VaR}_\alpha(L_i | L) = E[L_i | L = \text{VaR}_\alpha(L)]$ are applied on sub-sample with $L > 0$ only.

Example: credit risk with continuous losses (2)

From the R-code for a simulation run (standard sampling):

```
[1] ‘‘Sun Nov 12 15:57:03 2006’’
```

```
Importance sampling with tilting parameter: 0
```

```
Portfolio composition:
```

```
Exposures: 1 1 2 3 4 1 2 2
```

```
PDs: 0.05 0.025 0.01 0.005 0.001 0.1 0.025 0.025
```

```
Tilted PDs: 0.05 0.025 0.01 0.005 0.001 0.1 0.025 0.025
```

```
LGDs: 0.5 0.5 0.2 0.5 0.3 0.3 0.3 0.3
```

```
Std. deviations of LGDs: 0.25 0.25 0.2 0.25 0.229 0.229 0.229 0.229
```

```
Bandwidth: 0.03846664
```

```
# simulations: 250000
```

```
Conf. level: 0.999
```

```
Random seed: 1
```

Example: credit risk with continuous losses (3)

From the R-code for a simulation run (importance sampling):

```
[1] ‘‘Sun Nov 12 16:08:45 2006’’
```

```
Importance sampling with tilting parameter: 4.942
```

```
Portfolio composition:
```

```
Exposures: 1 1 2 3 4 1 2 2
```

```
PDs: 0.05 0.025 0.01 0.005 0.001 0.1 0.025 0.025
```

```
Tilted PDs: 0.3838 0.2328 0.068 0.8928 0.2736 0.3286 0.3321 0.3321
```

```
LGDs: 0.5 0.5 0.2 0.5 0.3 0.3 0.3 0.3
```

```
Std. deviations of LGDs: 0.25 0.25 0.2 0.25 0.229 0.229 0.229 0.229
```

```
Bandwidth: 0.09645164
```

```
# simulations: 250000
```

```
Conf. level: 0.999
```

```
Random seed: 3
```

Example: credit risk with continuous losses (4)

- Common approach to VaR contributions by practitioners:
 - Estimate $\text{VaR}_\alpha(L)$ by empirical VaR $L^{(\lceil \alpha T \rceil)}$.
 - Take $L_i^{(\lceil \alpha T \rceil)}$ as VaR contribution of loan i .
- **Pseudo contributions** as % of empirical VaR from two standard Monte-Carlo samples with different random seeds:
 - Random seed: 1
Practitioner's approach: 0 0 0 0 100 0 0 0
 - Random seed: 3
Practitioner's approach: 10.723 0 0 0 0 38.239 0 51.038

Example: credit risk with continuous losses (5)

Kernel estimation results from standard Monte Carlo sampling:

```
[1] ‘‘Sun Nov 12 15:57:03 2006’’
```

```
Importance sampling with tilting parameter: 0
```

```
Bandwidth: 0.03846664
```

```
# simulations: 250000
```

```
Conf. level: 0.999
```

```
Probability of zero loss in %: 77.921
```

```
[1] ‘‘Rel. VaR contributions as % of total VaR:’’
```

```
3.193 2.046 1.131 76.359 4.98 1.327 5.282 5.613
```

```
Sum of contributions as % of total VaR: 99.931
```

```
Estimated VaR: 2.521433
```

Example: credit risk with continuous losses (6)

Kernel estimation results from importance sampling:

```
[1] ‘‘Sun Nov 12 16:08:45 2006’’
```

```
Importance sampling with tilting parameter: 4.942
```

```
Bandwidth: 0.09645164
```

```
# simulations: 250000
```

```
Conf. level: 0.999
```

```
Probability of zero loss in %: 1.022
```

```
[1] ‘‘Rel. VaR contributions as % of total VaR:’’
```

```
2.795 2.044 2.38 71.157 7.312 2.76 5.436 5.679
```

```
Sum of contributions as % of total VaR: 99.562
```

```
Estimated VaR: 2.491170
```

Example: credit risk with continuous losses (7)

- Comparing variances of 10 repetitions of VaR contribution estimation for loan 7 by standard and importance sampling.

- Standard sampling:

Sim runs	25000	50000	100000	200000	400000	800000
Mean contrib	0.2146	0.1032	0.1196	0.1398	0.1581	0.1679
StDev/Mean	91.43%	108.38%	62.42%	35.21%	23.48%	23.09%

- Importance sampling:

Sim runs	25000	50000	100000	200000	400000	800000
Mean contrib	0.1920	0.1654	0.1491	0.1585	0.1531	0.1528
StDev/Mean	33.41%	29.69%	19.16%	9.74%	14.11%	9.95%

- Considerable reduction in variance of VaR contribution estimates although tilting parameter was not optimally chosen.

Conclusions

- VaR contributions are needed, e.g., for performance measurement or identifying risk concentrations.
- In credit risk practice, often VaR contributions have to be estimated from Monte Carlo samples.
- Kernel estimation approaches are more appropriate than naive approaches.
- Importance sampling significantly reduces variance of contribution estimates.
- Open issues:
 - Loans with identical risk characteristics should receive identical contributions.
 - Implementing Glasserman's two-step importance sampling for dependent L_i .