Interaction of Market and Credit Risk: An Analysis of Inter-Risk Correlation and Risk Aggregation

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Abstract

In this paper, we investigate the interaction between a credit portfolio and another risk type, which can be thought of as market risk. Combining Merton-like factor models for credit risk with linear factor models for market risk, we analytically calculate their inter-risk correlation and show how inter-risk correlation bounds can be derived. Moreover, we elaborate how our model naturally leads to a Gaussian copula approach for describing dependence between both risk types. In particular, we suggest estimators for the correlation parameter of the Gaussian copula that can be used for general credit portfolios. Finally, we use our findings to calculate aggregated risk capital of a sample portfolio both by numerical and analytical techniques.

1 Introduction

A core element of modern risk management and control is analyzing the *capital adequacy* of a financial institution, which is concerned with the assessment of the firm's required capital to cover the risks it takes. To this end, financial firms seek to quantify their overall risk exposure by aggregating all individual risks associated with different risk types or business units, and to compare this figure with a so-called *risk taking capacity*, defined as the total amount of capital as a buffer against potential losses.

Until now no standard procedure for risk aggregation has emerged but according to an industry survey of The Joint Forum [2], a widespread approach in the banking industry

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is *aggregation across risk types* where the marginal loss distributions of all relevant risk types are independently modelled from their dependence structure. In this paper, we concentrate on credit risk together with another risk type, which henceforth is referred to as market risk. Aggregation across risk types splits up into three steps:

- First, assign every individual risk position to a certain risk type.
- Second, calculate an aggregated measure for every risk type encompassing all business units by using separate, risk-type specific techniques and methodologies.
- Third, integrate all pre-aggregated risk figures of different risk types to obtain the overall capital number, henceforth called *aggregated risk capital*.

The easiest solution for the last step is simply to add up all pre-aggregated risk figures, which however is only a rough estimate of the bank-wide total risk exposure. Moreover, banks usually try to reduce their overall risk by accounting for diversification between different risk types because it allows them either to reduce their capital buffer (and thus expensive equity capital) or to increase their business volume (and thus to generate additional earnings). As a consequence thereof, return on equity and eventually shareholder value increases.

Hence, advanced approaches for risk aggregation begin with an analysis of the dependence structure between different risk types. Here, we combine a Merton-like factor model for credit risk with a linear factor model for market risk. Both models are driven by a set of (macroeconomic) factors $Y = (Y_1, \ldots, Y_K)$ where the factor weights are allowed to be zero so that a risk type may only depend on a subset of Y.

As an important measure of association, we start with an in-depth analysis of linear correlation between both risk types (henceforth referred to as *inter-risk correlation*). Our approach allows to derive closed-form expressions for inter-risk correlation in the case of normally distributed and heavy-tailed risk factors, providing valuable insight into inter-risk dependence of a credit risk portfolio in general. In particular, we give upper bounds for inter-risk correlation, which only depend on typical credit portfolio characteristics such as its asset correlation or rating structure.

Another very natural integration technique especially in the context of aggregation across risk types is based on copulas, see e.g. by Dimakos & Aas [5], Rosenberg & Schuermann [9]), Ward & Lee [10], or Böcker & Spielberg [4]. As a result of Sklar's theorem, copulas allow for a separate modelling of marginal distribution functions (second step above) on one hand and their dependence structure (third step above) on the other hand. However, the choice and parametrization of a copula is usually not straightforward, especially in the context of risk aggregation where reliable data are often difficult to obtain. We show that for large homogenous portfolios our model quite naturally leads to a Gaussian coupling model between both risk types and provide a simple estimator for the copula correlation parameter.

Finally, we perform a simulation study where we apply our findings to a test portfolio, for which aggregated risk capital is calculated by means of the copula technique as well as the well-known *square-root-formula* approach, which is a quite popular rule-of-thumb approximation for risk aggregation, see e.g. The Joint Forum [2] or Rosenberg & Schuermann [9]. While we observe that the square-root-formula seriously underestimates aggregated risk capital in the case of a Student t copula between market and credit risk, it seems to be a quite reasonable approximation if a Gaussian dependence structure is assumed.

2 Preliminaries: Modelling Credit and Market Risk

2.1 Factor Models for Credit Risk

To describe credit portfolio loss, we choose a classical structural model as it can be found e.g. in Bluhm, Overbeck and Wagner [3]. Within these models, a borrower's credit quality (and so his default behaviour) is driven by its *asset-value process*, or, more generally and in the case of unlisted customers, by a so-called "ability-to-pay" process. Consider a portfolio of n loans. Then, default of an individual obligor i is described by a Bernoulli random variable $L_i, i = 1, \dots, n$, with $\mathbb{P}(L_i = 1) = p_i$ and $\mathbb{P}(L_i = 0) = 1 - p_i$ where p_i is the obligor's *probability of default* within time period $[0, T]$ for $T > 0$. Following Merton's idea, counterparty i defaults if its asset value log-return A_i falls below some threshold D_i, sometimes referred to as *default point*, i.e.

$$L_i = \mathbb{1}_{\{A_i < D_i\}}\,, \qquad i = 1, \dots, n\,.$$

If we denote the exposure at default net recovery of an individual obligor by e_i, portfolio loss is finally given by

$$L^{(n)} = \sum_{i=1}^{n} e_i\, L_i\,. \tag{2.1}$$

Obviously, for a credit portfolio of n obligors, credit portfolio loss $L^{(n)}$ at time horizon T is driven by n realizations of the asset values A_i. Since for a typical credit portfolio n is very large, dimension reduction is called for. The technique elected in the context of credit risk modelling is usually a factor model approach. The following one is widely spread in financial firms and implemented in various software packages for credit risk.

Definition 2.1. [Normal factor model for credit risk] *Let* $Y = (Y_1, \ldots, Y_K)$ *be a* K*-dimensional random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, within the normal factor model we assume that each of the standardized asset value log-returns* $A_i,\ i = 1, \ldots, n$*, linearly depend on* $Y_k,\ k = 1, \ldots, K$*, as well as a standard normally distributed idiosyncratic factor (or noise term)* ε_i*, independent of all* Y_k*, i.e.*

$$A_i = \sum_{k=1}^{K} \beta_{ik} Y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\, \varepsilon_i\,, \qquad i = 1, \ldots, n\,, \tag{2.2}$$

with factor loadings satisfying $R_i^2 := \sum_{k=1}^{K} \beta_{ik}^2 \in [0,1]$ *describing the variance of* A_i *that can be explained by the systematic factors* $Y_k,\ k = 1, \ldots, K$*.*

For later usage we recall some properties of the normal factor model as can be found e.g. in Bluhm et al. [3] or McNeil, Frey and Embrechts [8], chapter 8.

Remark 2.2. (a) Equation (2.2) implies that the log-returns $(A_1, \ldots, A_n)$ are multivariate standard normally distributed with correlations

$$\rho_{ij} := \operatorname{corr}(A_i, A_j) = \sum_{k=1}^{K} \beta_{ik} \beta_{jk}\,, \qquad i, j = 1, \ldots, n\,, \tag{2.3}$$

the so-called *asset correlations* between A_i and A_j.
(b) The default point D_i of every obligor is related to its default probability p_i by

$$D_i = \Phi^{-1}(p_i)\,, \qquad i = 1, \ldots, n\,, \tag{2.4}$$

where Φ is the standard normal distribution function.
(c) The joint default probability of two obligors is given by

$$p_{ij} := \mathbb{P}(L_i = 1, L_j = 1) = \mathbb{P}(A_i \le D_i, A_j \le D_j) = \begin{cases} \Phi_{\rho_{ij}}(D_i, D_j)\,, & i \ne j\,, \\ p_i\,, & i = j\,, \end{cases} \tag{2.5}$$

where $\Phi_{\rho_{ij}}$ denotes the bivariate standard normal distribution function with correlation ρ_{ij} given by (2.3). Moreover, the *default correlation* between two different obligors is given by

$$\operatorname{corr}(L_i, L_j) = \frac{p_{ij} - p_i\, p_j}{\sqrt{p_i(1 - p_i)\, p_j(1 - p_j)}}\,, \qquad i, j = 1, \ldots, n\,. \tag{2.6}$$

(d) Conditional on a realization $y = (y_1, \dots, y_K)$ of the factors Y, defaults of different obligors are independent. Moreover, the *conditional default probability* is given by

$$
\begin{aligned}
p_i(y) &= \mathbb{P}(L_i = 1 | \, Y = y) \\
&= \mathbb{P}\left(\sum_{k=1}^{K} \beta_{ik}\, y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\, \varepsilon_i \le D_i \right) \\
&= \Phi\left(\frac{D_i - \sum_{k=1}^{K} \beta_{ik}\, y_k}{\sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}} \right).
\end{aligned}
$$

□

A strong assumption of the model above is the multivariate normal distribution of the factor variables Y_k, $k = 1, \dots, K$, and thus of the asset value log-returns A_i. It is well known that the normal distribution has very light tails and therefore may seriously underestimate large fluctuations of the (macroeconomic) factors, eventually leading to model risk of the normal factor model for credit risk.

A generalization allowing for heavier tails as well as a stronger dependence between different counterparties is the class of normal variance mixture distributions, where the covariance structure of the A_i is disturbed by means of a positive mixing variable W, see e.g. McNeil et al. [8], section 3.2. A particular interesting model is the following one, confer also Kostadinov [7]:

Definition 2.3. [Shock model for credit risk] *Let $(Y_1, \dots, Y_K)$ be a K-dimensional random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, the shock model assumes that each of the standardized asset value log-returns A_i, $i = 1, \dots, n$, can be written as*

$$
A_i = W_L \cdot \sum_{k=1}^{K} \beta_{ik} Y_k + W_L \cdot \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\, \varepsilon_i \, , \qquad i = 1, \dots, n \, , \tag{2.7}
$$

where $W_L = \sqrt{\nu / S_{\nu_L}}$ and S_{ν_L} is a $\chi^2_{\nu_L}$ distributed random variable with ν_L degrees of freedom, independent of all Y_k as well as the idiosyncratic factor ε_i. The other quantities are as in Definition 2.1.

The mixing variable W_L can be interpreted as a "global shock" driving the variance of all factors. Such an overarching shock may occur from political distress, severe economic recession or some natural disaster.

We conclude this section with some general remarks about the shock model for credit risk (see again Bluhm et al. [3] and McNeil et al. [8] as well as references therein).

Remark 2.4. (a) In general, if $(X_1, \ldots, X_n)$ is a standardized multinormal vector with covariance matrix R and S_{ν_L} is a chi-square variable with ν_L degrees of freedom, then $(X_1, \ldots, X_n)/\sqrt{S_{\nu_L}/\nu_L}$ has a multivariate t-distribution with correlation matrix R and ν_L degrees of freedom. Hence, from (2.2) and (2.7) it follows for the shock model for credit risk that the vector of standardized asset value log-returns $(A_1, \ldots, A_n)$ is t-distributed with ν_L degrees of freedom, in particular, it has the same asset correlation ρ_{ij} as the normal factor model given by equation (2.3).
(b) The default point D_i is linked to the obligor's default probability by

$$D_i = t_{\nu_L}^{-1}(p_i)\,, \qquad i = 1, \ldots, n\,, \tag{2.8}$$

where t_{ν_L} is the Student t distribution function with ν_L degrees of freedom.
(c) The joint default probability p_{ij} for two obligors can be written as

$$p_{ij} = t_{\nu_L;\rho_{ij}}(D_i, D_j)\,, \qquad i \neq j\,, \tag{2.9}$$

where $t_{\nu_L;\rho_{ij}}$ denotes the standard bivariate Student t distribution function with correlation ρ_{ij} given by (2.3) and degree of freedom parameter ν_L. □

2.2 Factor Models for Market Risk

Market risk is related to a bank's potential loss associated with its trading activities. We assume that it is already pre-aggregated so that losses can be approximately described by a one-dimensional random variable Z, which can be thought of as the bank-wide, aggregated profit and loss (P/L) distribution due to changes in some market variables, such as interest rates, foreign exchange rates, equity prices or the value of commodities.

Similarly as we did for credit risk, we try to explain fluctuations of the random variable Z by means of (macroeconomic) factors $Y_k, k = 1, \ldots, K$. If the pre-aggregated P/L can be described by a normal distribution, the following factor model is a sensible choice and can be used for risk aggregation. Even if this assumption does not hold exactly, it can be used as an important approximation for investigating inter-risk dependencies (we use the convention that losses correspond to positive values of Z.)

Definition 2.5. [Normal factor model for market risk] *Let Z be a random variable describing pre-aggregated market risk P/L. Then, the linear factor model is given by*

$$Z = -\sigma \left(\sum_{k=1}^{K} \gamma_k Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\; \eta \right), \tag{2.10}$$

satisfying $\sum_{k=1}^{K} \gamma_k^2 \in [0, 1]$. Furthermore, $Y_k,\ k = 1, \ldots, K$, are the same (macroeconomic) factors as in the credit risk model of Definition 2.1, η is a standard normally distributed *idiosyncratic factor, independent of the Y_k and the credit model's idiosyncratic factors ε_i, $i = 1, \ldots, n$, and σ is the standard deviation of Z.*

Clearly, for an actively managed market portfolio the idiosyncratic factor η is more important as for a unmanaged portfolio (e.g. an index of stocks). As a matter of fact, portfolio managers are paid owing to their skills to achieve best possible portfolio performance that is independent of some macroeconomic indicators.

Note that both in Definition 2.1 of the credit factor model as well as above, the factor loadings β_{ik} and γ_k, respectively, are allowed to be zero. For instance, Y_k can be relevant for credit but not for market by setting $\gamma_k = 0$.

In order to account for possible heavy tails in the market risk P/L, we again rely on the global shock approach already used for credit risk.

Definition 2.6. [Shock model for market risk] *Let Z be a random variable describing market risk P/L. Then, the shock model is given by*

$$Z = -\sigma \left(W_Z \cdot \sum_{k=1}^{K} \gamma_k Y_k + W_Z \cdot \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2 \, \eta} \right), \tag{2.11}$$

where $W = \sqrt{\nu_Z / S_{\nu_Z}}$ and S_{ν_Z} is a $\chi^2_{\nu_Z}$ distributed random variable with ν_Z degrees of freedom, independent of all Y_k as well as the idiosyncratic factor η. The other quantities are as in Definition 2.5

3 Inter-Risk Correlation

3.1 Normal Factor Model Approach

The proposed models shall now be used to investigate the dependence between credit risk $L^{(n)}$ and market risk Z, introduced by $Y = Y_1, \ldots, Y_K$. Let us start with the linear correlation, which is defined as

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\operatorname{cov}(L^{(n)}, Z)}{\sqrt{\operatorname{var}(L^{(n)})}\sqrt{\operatorname{var}(Z)}} \,. \tag{3.1}$$

Although linear correlation only describes linear dependence between different random variables, it is a very popular and important concept in finance, frequently used both by practitioners and academics. Moreover, since we calculate expressions for linear correlation in closed-form, we are able to analytically investigate the linear dependence structure between market and credit risk.

We begin with the normal factor models for credit and market risk. Here as well as for all subsequent results, all proofs are given in the appendix.

Theorem 3.1 (Inter-risk correlation for the normal factor model). *Suppose that credit portfolio loss $L^{(n)}$ and market risk Z are described by the normal factor models of Definitions 2.1 and 2.5, respectively. Then, linear correlation between $L^{(n)}$ and Z is given by*

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\sum_{i=1}^{n} r_i \, e_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \operatorname{var}(L^{(n)})}}\,, \tag{3.2}$$

where

$$r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^{K} \beta_{ik} \gamma_k\,, \qquad i = 1, \ldots, n,$$

and

$$\operatorname{var}(L^{(n)}) = \sum_{i,j=1}^{n} e_i \, e_j \, (p_{ij} - p_i \, p_j)$$

is the variance of the credit portfolio. Furthermore, D_i is the default point (2.4), *and p_{ij} the joint default probability* (2.5).

Note that r_i may become negative if (some) factors weights β_{ik} and γ_k have different sign. Therefore, in principal, also negative inter-risk correlations can be achieved in our model. Moreover, in (3.2) the term $e_i \, e^{-D_i^2/2}$ can be interpreted as a kind of rating-adjusted exposure. For instance, a relatively low default probability of debtor i corresponds to a relatively small value of $e^{-D_i^2/2}$. As a consequence thereof, for two obligors with equal exposure size e_i, the one with the better rating has less impact on the inter-risk correlation as the low-rated creditor.

The fact that $\operatorname{corr}(L^{(n)}, Z)$ linearly depends on the correlation r_i and thus on the factor loadings γ_k, implies the following Proposition.

Proposition 3.2 (Inter-risk correlation bounds for the normal factor model). *Suppose that credit portfolio loss $L^{(n)}$ is quantified by the factor model of Definition 2.1 and that market risk Z follows the linear model of Definition 2.5, however, with unknown factor loadings γ_k, $k = 1, \ldots, K$. Then, inter-risk correlation is bounded by*

$$|\operatorname{corr}(L^{(n)}, Z)| \;\leq\; \frac{\sum_{i=1}^{n} e_i \sqrt{\sum_{k=1}^{K} \beta_{ik}^2} \, \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \operatorname{var}(L^{(n)})}}\,. \tag{3.3}$$

Therefore, solely based on the parametrization of the normal credit factor model, bounds for the inter-risk correlation can be derived. These bounds then hold for all market risk portfolios described by Definition 2.5. Furthermore, as $R_i^2 = \sum_{k=1}^{K} \beta_{ik}^2$ is that part of the variance of A_i which can be explained by the factors Y_k, it follows from (3.3) that inter-risk correlation bounds are linearly increasing with R_i. This is also intuitively clear because with increasing R_i^2, credit portfolio loss is more and more dominated by the systematic factors $Y = (Y_1, \dots, Y_K)$, which by construction drive the inter-risk dependence with market risk.

3.2 Shock Model Approach

We now use the shock models of Definition 2.3 and 2.6 to determine inter-risk correlation similarly to Theorem 3.1.

Theorem 3.3 (Inter-risk correlation for the shock factor model)**.** *Suppose that credit portfolio loss $L^{(n)}$ and market risk Z are described by the shock factor models of Definitions 2.3 and 2.6.*
(1) (Independent shocks). *If shocks in credit and market risk are driven by different independent shock variables W_L and W_Z with degrees of freedom ν_L and ν_Z, respectively, linear correlation between $L^{(n)}$ and Z is given by*

$$\operatorname{corr}(L^{(n)}, Z) = \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \frac{\sum_{i=1}^n e_i\, r_i \left(1 + \frac{D_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi \operatorname{var}(L^{(n)})}}\,. \tag{3.4}$$

(2) (Common shocks). *If shocks in credit and market risk are driven by the same shock variable $W := W_L = W_Z$ with $\nu := \nu_L = \nu_Z$ degrees of freedom, linear correlation between $L^{(n)}$ and Z is given by*

$$\operatorname{corr}(L^{(n)}, Z) = \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, \frac{\sum_{i=1}^n e_i\, r_i \left(1 + \frac{D_i^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi \operatorname{var}(L^{(n)})}}\,. \tag{3.5}$$

In both cases,

$$r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^K \beta_{ik}\gamma_k\,, \qquad i = 1, \dots, n,$$

and

$$\operatorname{var}(L^{(n)}) = \sum_{i,j=1}^n e_i\, e_j\, (p_{ij} - p_i\, p_j)$$

is the variance of the credit portfolio. Furthermore, D_i and p_{ij} are given by (2.8) *and* (2.9), *respectively, with degree of freedom ν_L in (1) and ν in (2).*

Analogously to the normal factor model, inter-correlation bounds can be derived also for the shock model.

Proposition 3.4 (Inter-risk correlation bounds for the shock model)**.** *Suppose that credit portfolio loss $L^{(n)}$ is quantified by the shock model of Definition 2.3 and that market risk Z follows the shock model of Definition 2.6, with unknown factor loadings $\gamma_k,\ k = 1, \dots, K$.*
(1) *For the independent shock model, inter-risk correlation is bounded by*

$$|\mathrm{corr}(L^{(n)}, Z)| \quad \leq \quad \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \frac{\sum_{i=1}^n e_i \sqrt{\sum_{k=1}^K \beta_{ik}^2}\left(1 + \frac{D_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi\,\mathrm{var}(L^{(n)})}}\,. \tag{3.6}$$

(2) *For the common shock model, inter-risk correlation is bounded by*

$$|\mathrm{corr}(L^{(n)}, Z)| \quad \leq \quad \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, \frac{\sum_{i=1}^n e_i \sqrt{\sum_{k=1}^K \beta_{ik}^2}\left(1 + \frac{D_i^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi\,\mathrm{var}(L^{(n)})}}\,. \tag{3.7}$$

A relevant example is where the credit risk portfolio is described by a normal factor model, whereas heavy-tails are assumed for the market risk distribution.

Example 3.5. [Normal factor model for credit risk, shock model for market risk]
Assume a normal factor model for credit risk according to Definition 2.1 and a shock model with ν_Z degrees of freedom for market risk specified by Definition 2.6. This is equivalent to the independent shock model of Theorem 3.3 (1) with $\nu_Z \to \infty$ and inter-risk correlation and its bound can formally be obtained from (3.4) and (3.6), respectively, when $(1 + \frac{D_i^2}{\nu_L})^{-\frac{\nu_L}{2}}$ is substituted by $e^{-D_i^2/2}$ and p_{ij} is given by (2.5). □

4 An Application to One-Factor Models

4.1 Common One-Factor Models for Market And Credit Risk

A particular important example in the context of credit risk modelling is the case of a large homogenous portfolio (LHP).

Let us start with a homogenous portfolio for which we define that $e_i = e$, $p_i = p$, $\beta_{ik} = \beta_k$ for $i = 1, \dots, n$, and $k = 1, \dots, K$, and expression (2.2) for the general factor model can be transformed into a one-factor model by setting $\rho := \sum_{k=1}^K \beta_k^2$ and $\tilde{Y} := \left(\sum_{k=1}^K \beta_k Y_k\right)/\sqrt{\rho}$:

$$A_i = \sqrt{\rho}\,\tilde{Y} + \sqrt{1-\rho}\,\varepsilon_i\,, \tag{4.1}$$

where $\tilde{Y}$ is standard normally distributed and independent of ε_i, and ρ is the *uniform asset correlation* within the credit portfolio. If we now additionally increase the number of counterparties in the portfolio by $n \to \infty$, then the relative portfolio loss satisfies[1]

$$\frac{L^{(n)}}{n\,e} \overset{\text{a.s.}}{\to} \Phi\left(\frac{D - \sqrt{\rho}\,\tilde{Y}}{\sqrt{1-\rho}}\right) =: L\,, \qquad n \to \infty\,, \tag{4.2}$$

where $D = \Phi^{-1}(p)$ and $n\,e$ is the total exposure of the credit portfolio. Often L is used as an approximative loss variable for large and almost homogeneous portfolios. For later usage we note that the variance of L is given by

$$\text{var}(L) = p_{12} - p^2 \tag{4.3}$$

with $p_{12} = \Phi_\rho(D, D)$.

In the case of the shock model, the LHP approximation reads

$$\frac{L^{(n)}}{n\,e} \overset{\text{a.s.}}{\to} \Phi\left(\frac{D/W_L - \sqrt{\rho}\,\tilde{Y}}{\sqrt{1-\rho}}\right) =: L_{\nu_L}\,, \qquad n \to \infty\,,$$

where now $D = t_{\nu_L}^{-1}(p)$. The variance is given by (4.3) with $p_{12} = t_{\nu_L;\rho}(D, D)$.

We now extend the one-factor model for credit risk to market risk so that the systematic change of the market risk P/L is described only by the single factor $\tilde{Y}$. To achive this, we set $\tilde{Y} := \left(\sum_{k=1}^K \beta_k Y_k\right) / \sqrt{\rho}$ and

$$\tilde{\eta} := \frac{1}{\sqrt{1-\tilde{\gamma}^2}} \left[\sum_{k=1}^K \left(\gamma_k - \frac{\tilde{\gamma}}{\sqrt{\rho}}\,\beta_k\right) Y_k + \sqrt{1 - \sum_{k=1}^K \gamma_k^2}\,\eta\right]$$

with

$$\tilde{\gamma} := \frac{1}{\sqrt{\rho}} \sum_{k=1}^K \beta_k \gamma_k\,. \tag{4.4}$$

Then, we obtain the formal identities

$$Z = -\sigma\left(\tilde{\gamma}\,\tilde{Y} + \sqrt{1-\tilde{\gamma}^2}\,\tilde{\eta}\right) \tag{4.5}$$

and

$$Z = -\sigma\,W_Z\left(\tilde{\gamma}\,\tilde{Y} + \sqrt{1-\tilde{\gamma}^2}\,\tilde{\eta}\right) \tag{4.6}$$

for the normal factor model (2.10) and for the shock model (2.11), respectively. In both cases, $\tilde{\eta}$ is standard normally distributed and independent of $\tilde{Y}$. Moreover, Z in (4.5) is

[1] Actually, there are less restrictive conditions for the exposures e_i and the individual default variables L_i under which the LHP approximation still holds, see e.g. in Bluhm et al. [3], section 2.5.1.

normally distributed with zero mean and variance $\mathrm{var}(Z) = \sigma^2$, whereas in (4.6) it follows a t-distribution with ν_Z degrees of freedom.

While the one-factor weight $\sqrt{\rho}$ for the credit portfolio depends only on the β_k, the one-factor weight $\tilde{\gamma}$ for market risk given by (4.4) is a function of $\beta_k \gamma_k$. In particular, in order to obtain non-vanishing systematic market risk within the one-factor model, both risk types have to share at least one common factor.

4.2 One-Factor Inter-Risk Correlation

Very instructive examples regarding the inter-risk correlation described in section 3 can be obtained in the case of one-factor models for which the previous results simplify considerably. Such a one-factor framework is very appropriate to explore general characteristics and systematic behaviour of inter-risk correlation.

Normal Factor Model Approach. Instead of (3.2) we now obtain

$$\mathrm{corr}(L^{(n)}_{\mathrm{hom}}, Z) = \frac{\sqrt{n}\, r\, e^{-D^2/2}}{\sqrt{2\pi}\sqrt{p_{12}(n-1) + p(1-np)}},$$

where $D = \Phi^{-1}(p)$ is the default point, $p_{12} = \Phi_\rho(D, D)$ is the joint default probability within the homogenous portfolio, and $r = \mathrm{corr}(Z, A_i) = \sqrt{\rho}\,\tilde{\gamma} = \sum_{k=1}^{K} \beta_k \gamma_k$. If the credit portfolio is not only homogenous but also very large, we obtain the following LHP approximation for the inter-risk correlation:

$$\mathrm{corr}(L, Z) = \frac{r\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}}\,. \tag{4.7}$$

Given the uniform asset correlation $\rho = \sum_{k=1}^{K} \beta_k^2$ of a credit portfolio, it follows from (4.4) that $|\tilde{\gamma}| \leq 1$, and thus $|r| \leq \sqrt{\rho}$, implying the bounds

$$|\mathrm{corr}(L, Z)| \leq \frac{\sqrt{\rho}\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}} =: B_{\mathrm{LHP}}(p, \rho)\,. \tag{4.8}$$

Shock Model Approach. The LHP approximation for the inter-risk correlation in the case of independent shocks yields

$$\mathrm{corr}(L^{(n)}, Z) = \sqrt{\frac{\nu_Z - 2}{2}}\, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\left(\frac{\nu_Z}{2}\right)}\, \frac{r\left(1 + \frac{D^2}{\nu_L}\right)^{-\frac{\nu_L}{2}}}{\sqrt{2\,\pi\,(p_{12} - p^2)}}\,, \tag{4.9}$$

whereas for the common shock model we obtain

$$\mathrm{corr}(L^{(n)}, Z) = \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu - 1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\, \frac{r\left(1 + \frac{D^2}{\nu}\right)^{\frac{1-\nu}{2}}}{\sqrt{2\,\pi\,(p_{12} - p^2)}}\,, \tag{4.10}$$

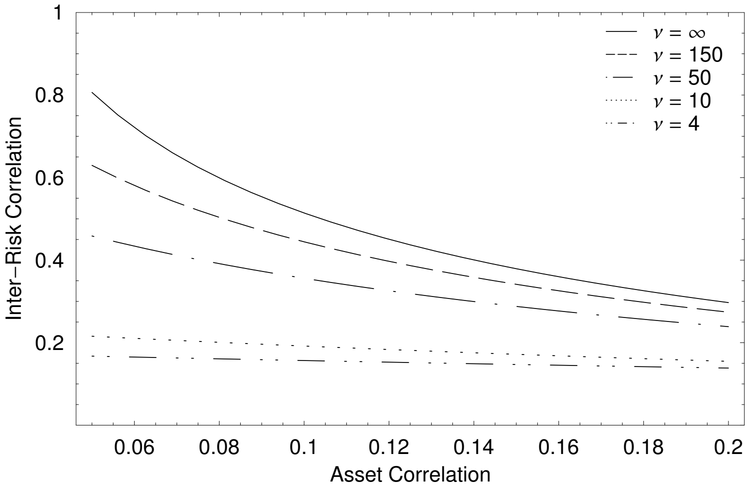

Figure 4.1. *LHP approximations of inter-risk correlation for* $r = 0.2$ *as a function of the asset correlation* ρ *according to the normal factor model (equation* (4.7)*) and the common shock model (equation* (4.10)*). The average default probability is assumed to be* $p = 0.002$*.*

where $D = t_\nu^{-1}(p)$ is the default point, $p_{12} = t_{\nu;\rho}(D, D)$ is the joint default probability within the homogenous portfolio, and $r = \text{corr}(Z, A_i) = \sqrt{\rho}\,\tilde{\gamma}$. Similarly as for the LHP approximation of normal factor model, bounds for the inter-risk correlation can be obtained from (4.9) and (4.10) together with $|r| \leq \sqrt{\rho}$. In the special case that $\nu_L = \nu_Z = \nu$ it follows from (4.9) and (4.10) that the assumption of one single common shock increases inter-risk correlation by a factor of $\sqrt{1 + \frac{D^2}{\nu}}$ compared to the independent shock model. For typical values of p this factor lies in a range of about 1.0–2.0.

The LHP approximations for inter-risk correlation of the shock models given by (4.9) and (4.10) can also be compared to the LHP approximation (4.7) of the normal factor model. This is useful to find out how global (macroeconomic) shocks affect inter-risk correlation. For this purpose, Table 4.1 as well as Figures 4.1 and 4.2 compare the inter-risk correlation and its upper bound for the common shock model with the outcome of the normal factor model. One can see that the common shock model yields—particularly for small asset correlations—lower inter-risk correlation as well as inter-risk correlation bounds than the normal factor model. In the case of the independent shock model, the spread between the normal inter-risk correlation and the shocked inter-risk correlation would be even higher.

Needless to say, the one-factor asset correlation ρ is a popular parameter in the context of credit portfolio modelling. It is often used as a single-number summary in order to compare dependence structures within different credit portfolios, and it plays an important role also in the calculation formula for regulatory capital charges according to the *internal-*

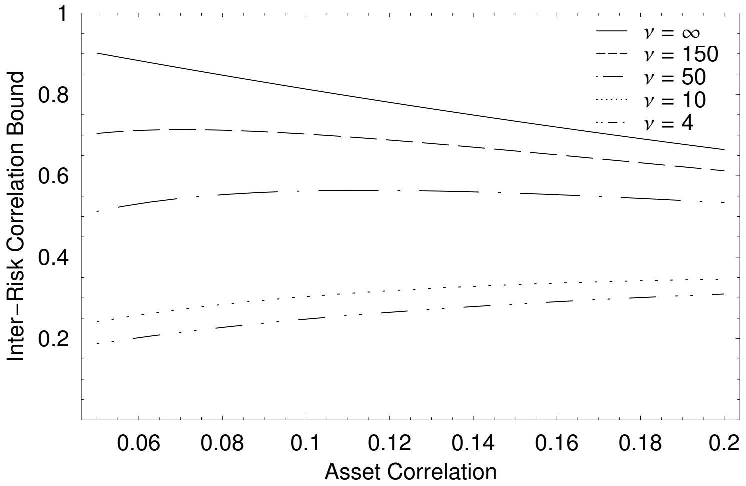

Figure 4.2. *LHP approximations of the inter-risk correlation bound as a function of the asset correlation ρ according to the normal factor model (equation (4.8)) and the common shock model (equation (4.10) with $r = \sqrt{\rho}$). The average default probability is assumed to be $p = 0.002$.*

ρ	Normal Model	Common Shock Model		
	$\nu = \infty$	$\nu = 4$	$\nu = 10$	$\nu = 50$
		$p = 0.002$		
5 %	0.81 (0.90)	0.17 (0.19)	0.22 (0.24)	0.46 (0.51)
10 %	0.51 (0.81)	0.16 (0.25)	0.19 (0.30)	0.36 (0.56)
15 %	0.38 (0.73)	0.15 (0.28)	0.17 (0.33)	0.29 (0.56)
20 %	0.30 (0.66)	0.14 (0.31)	0.15 (0.35)	0.24 (0.53)
		$p = 0.02$		
5 %	0.85 (0.95)	0.27 (0.31)	0.37 (0.42)	0.62 (0.70)
10 %	0.57 (0.90)	0.25 (0.40)	0.33 (0.52)	0.48 (0.76)
15 %	0.44 (0.86)	0.24 (0.46)	0.29 (0.57)	0.39 (0.76)
20 %	0.37 (0.82)	0.22 (0.50)	0.27 (0.59)	0.33 (0.75)

Table 4.1: LHP approximation for inter-risk correlation for the normal factor model (4.7) and the common shock model (4.10) using $r = 0.2$ but different values for p and asset correlation ρ. The values in brackets indicate upper inter-risk correlation bounds for which $r = \sqrt{\rho}$.

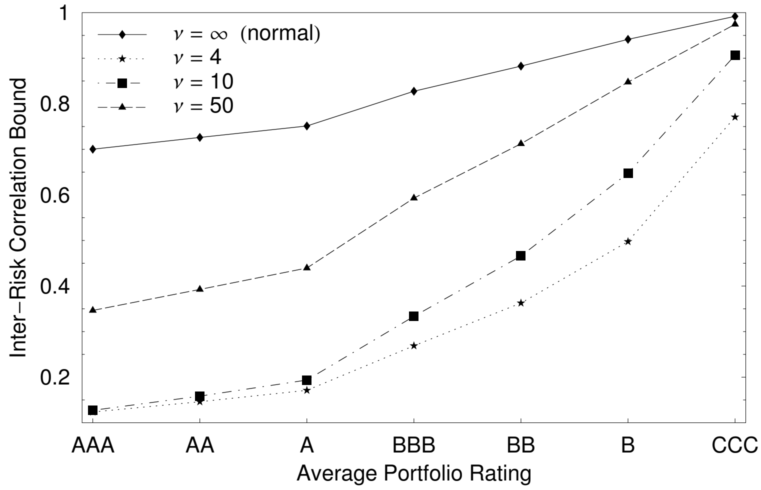

Figure 4.3. *LHP approximations of inter-risk correlation bound as a function of the average portfolio rating according to the normal factor model (equation* (4.8)*) and the common shock model (equation* (4.10) *with* $r = \sqrt{\rho}$*). The average asset correlation is assumed to be* $\rho = 10$ %.

ratings-based (IRB) approach of Basel II [1]. Equations (4.8)-(4.10) now show that ρ is a very important parameter also beyond the credit portfolio itself as it determines the maximum possible inter-risk correlation of credit risk, see again Figure 4.2 where inter-risk correlation bounds are plotted as a function of ρ[2].

Similarly, for a fixed uniform asset correlation ρ, inter-risk correlation and its bound depend on the average default probability p and thus on the average rating of the credit portfolio. This is depicted in Figure 4.3 where LHP inter-risk correlation as well as the corresponding upper bounds are plotted as a function of the average portfolio rating. One can see that an improvement of the average portfolio rating structure decreases inter risk correlation (as well as its bounds) and thus tends to result in a lower volatility of the total portfolio of market and credit.

Even if actual credit portfolios are rarely exactly homogenous, the derived LHP approximations can be used as a practical approximation for the upper inter-risk correlation bound. Let us consider the normal factor model and so equation (4.8). For a loss distribution of a general credit portfolio (obtained for instance by Monte Carlo simulation) with expected loss μ, standard deviation ς, and total exposure e_{tot}, estimators $\hat{p}$ and $\hat{\rho}$ for p and ρ, respectively, can be (numerically) determined by comparing the expected loss and

[2]Note that ρ enters $|\text{corr}(L, Z)|$ not only directly by $\sqrt{\rho}$ but also indirectly via the joint default probability $p_{12} = \Phi_\rho(\Phi^{-1}(p), \Phi^{-1}(p))$. This implies that $|\text{corr}(L, Z)| \neq 0$ for $\rho \to 0$.

the variance of the simulated portfolio with those of an LHP (moment matching), i.e.

$$\mu \;=\; e_{\text{tot}}\,\hat{p} \tag{4.11}$$

and

$$\begin{aligned} \varsigma^2 \;&=\; e_{\text{tot}}^2\,(p_{12}-\hat{p}^2) \\ &=\; e_{\text{tot}}^2\left[\Phi_{\hat{\rho}}(\Phi^{-1}(\hat{p}),\Phi^{-1}(\hat{p}))-\hat{p}^2\right]. \end{aligned} \tag{4.12}$$

From (4.8) we then obtain the following moment estimator for the upper inter-risk correlation bound

$$\widehat{B}_{\text{LHP}}(\hat{p},\hat{\rho}) = \frac{e_{\text{tot}}}{\varsigma}\,\frac{\sqrt{\hat{\rho}}\,\exp\left[-\frac{1}{2}\big(\Phi^{-1}(\hat{p})\big)^2\right]}{\sqrt{2\pi}}\,, \tag{4.13}$$

which of course can be compared to the exact bound given in Proposition 3.2. For instance, for the credit test portfolio described in Appendix 6.1 we have $e_{\text{tot}}/\varsigma = 92.41$, $\hat{p} = 0.54$ %, $\hat{\rho} = 23.31$ %, and (4.13) yields $\widehat{B}_{\text{LHP}} = 0.69$. In contrast, the exact bound for the inter-risk correlation (3.3) evaluates to 0.57.

5 Risk Aggregation

As we already mentioned in the introduction, the estimation of aggregated economic capital is a key element both for regulatory and bank internal purposes. Usually economic capital is defined as the unexpected loss, more precisely, for a given risk type X with strictly increasing and continuous distribution function $F(x) = \mathbb{P}(X \leq x)$, economic capital at confidence level κ is given by

$$\text{EC}(\kappa) = F^{-1}(\kappa) - \mathbb{E}(X)$$

where $F^{-1}(\cdot)$ is the inverse of $F(\cdot)$ and $\mathbb{E}(X)$ is the expectation of X.

The proposed model here allows to compare the four most important approaches for risk aggregation that are used in practice or discussed in the literature.

First, most straight forward is clearly the simple summation of pre-aggregated risk figures. Though it typically overestimates total risk, it is still used in practice. Second, aggregated risk capital can be obtained by a joint Monte Carlo simulation of the (macroeconomic) factors $Y_1,\ldots,Y_K$ as well as the idiosyncratic factors ε_i and η entering both (2.2) and (2.10). Finally, the last two techniques to mention are the copula approach and the square-root-formula approach, which shall be considered in greater detail below. In doing so, we henceforth restrict ourselves to the normal factor model approaches given by Definitions 2.1 and 2.5.

Square-Root-Formula Approach. Though mathematically justified only in the case of elliptically (e.g. multivariate normally) distributed risk types, this approach is often used in practise because risk-type aggregation can be achieved without simulation. In our bivariate case of a credit portfolio $L^{(n)}$ and (pre-aggregated) market portfolio Z, the square-root formula reads

$$\mathrm{EC}_\kappa(L^{(n)} + Z) = \sqrt{\mathrm{EC}_\kappa(L^{(n)})^2 + \mathrm{EC}_\kappa(Z)^2 + 2\,\mathrm{corr}(L^{(n)}, Z)\,\mathrm{EC}_\kappa(L^{(n)})\,\mathrm{EC}_\kappa(Z)}\,,$$

where $\mathrm{corr}(L^{(n)}, Z)$ is the inter-risk correlation (3.2).

Copula Aggregation Approach. Basically, a d-dimensional copula C is a d-dimensional distribution function on $[0,1]^d$. The relevance of distributional copulas for risk integration is mainly because for a given copula C and marginal distribution functions $F_1, \dots, F_d$, the joint distribution can be obtained via

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \tag{5.1}$$

Therefore, if one has specified marginals for each risk type together with an appropriate copula, total aggregate loss can be obtained numerically or by Monte Carlo simulations.

A remarkable feature of the suggested model here is that under certain conditions it can be easily interpreted as a Gaussian coupling model between market and credit risk.

Proposition 5.1 (Normal one-factor model and Gaussian copula)**.** *Consider the one-factor representation for credit and market risk given by* (4.1) *and* (4.5)*, respectively. Then*

(1) *both risk types are coupled by a Gaussian copula with parameter* $\tilde{\gamma}$ *given by* (4.4)*.*

(2) *the copula parameter* $\tilde{\gamma}$ *and the inter-risk correlation* $\mathrm{corr}(L, Z)$ *are related by*

$$\tilde{\gamma} = \frac{\mathrm{corr}(L, Z)}{B_{\mathrm{LHP}}} \tag{5.2}$$

where B_{LHP} *is the LHP approximation* (4.8) *for the inter-risk correlation bound.*

It follows from (5.2) that the absolute value of the inter-correlation between credit and market risk is always below the absolute value of the copula parameter $\tilde{\gamma}$. Furthermore, maximum inter-risk correlation corresponds to $\tilde{\gamma} = 1$ for which market risk is completely determined by one single risk factor without having any idiosyncratic component, cf. equation (4.5). A numerical example for (5.2) is given in Table 5.2.

Particularly important for practical applications is the question how the Gaussian copula parameter can be estimated for general credit portfolios. Note that in this case Proposition 5.1 (1) is not directly applicable because β_k in (4.4) is only defined for a

$\tilde{\gamma}$	0.0	0.2	0.4	0.6	0.8	1.0
$\text{corr}(L, Z)$	0.0	0.15	0.29	0.44	0.59	0.73

Table 5.2: Relation between inter-risk correlation $\text{corr}(L, Z)$ and Gaussian copula parameter $\tilde{\gamma}$ according to (5.2) for $p = 0.002$ and $\rho = 15$ %.

homogenous portfolio. However, we can extend the LHP credit portfolio approximation—as given by (4.11) and (4.12)—to the *aggregated* one-factor risk model including also market risk by matching the inter-risk correlations. Then, using Proposition 5.1 (2) we arrive at the following general estimator for the copula parameter $\tilde{\gamma}$,

$$\widehat{\tilde{\gamma}}_1 = \frac{\widehat{\text{corr}}(L, Z)}{\widehat{B}_{\text{LHP}}} \tag{5.3}$$

where $\widehat{B}_{\text{LHP}}$ is the moment estimator for the LHP inter-risk correlation bound given by (4.13) and

$$\widehat{\text{corr}}(L, Z) = \text{corr}(L^{(n)}, Z)$$

with $\text{corr}(L^{(n)}, Z)$ as in (3.2).

An alternative estimator for $\tilde{\gamma}$ can be constructed by applying the right-hand side of (5.2) directly to a non-homogenous portfolio without introducing a one-factor approximation before. In this case it follows together with (3.2) and (3.3) that

$$\widehat{\tilde{\gamma}}_2 = \frac{\sum_{i=1}^{n} \sum_{k=1}^{K} \beta_{ik}\, \gamma_k\, e_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sum_{i=1}^{n} e_i \sqrt{\sum_{k=1}^{K} \beta_{ik}^2}\, \exp\left(-\frac{1}{2} D_i^2\right)}\,. \tag{5.4}$$

We now illustrate our findings obtained so far by means of the sample portfolio of credit and market risk described in Appendix 6.1. For the estimators above as well as the inter-risk correlation (3.2) we then obtain $\widehat{\tilde{\gamma}}_1 = 0.32$, $\widehat{\tilde{\gamma}}_2 = 0.39$, and $\text{corr}(L^{(n)}, Z) = 0.22$. These parameters can now be used for risk aggregation as described before. Results for aggregated economic capital at different confidence levels κ are are summarized in Table 5.3. Some remarks are appropriate. In the first two rows of Table 5.3, stand-alone credit risk and market risk were calculated by the general models of Definition 2.1 and 2.5, respectively. These figures were directly used in the square-root formula approach. In contrast, for the copula aggregation the marginal distribution function for credit risk was first approximated by a one-factor model using moment matching (4.11) and (4.12). Finally, the copula parameter was estimated via the moment estimator $\widehat{\tilde{\gamma}}_1$.

In this particular example the square-root formula seems to be a reasonable proxy when economic capital at high confidence levels is considered. The observation that the copula approach gives larger total economic capital at high confidence levels is partially

EC	$\kappa = 0.9$	$\kappa = 0.99$	$\kappa = 0.999$	$\kappa = 0.9998$
Credit	0.16	0.87	1.91	2.68
Market	0.23	0.42	0.56	0.64
Sum	0.39	1.29	2.47	3.32
Square-root formula ($\text{corr}(L^{(n)}, Z) = 0.22$)	0.31	1.04	2.10	2.89
Copula approach ($\tilde{\gamma} = 0.32$)	0.32	1.03	2.20	3.16
Simulation approach	0.32	1.04	2.09	2.89

Table 5.3: Aggregated economic capital in EUR bn for different confidence levels κ obtained by the four aggregation methods described in the text.

due to the fact that the one-factor approximation for the credit risk marginal obtained by moment matching is heavier-tailed than the simulated credit risk distribution. Therefore, if one is mainly interested in economic capital at high confidence levels, an alternative approximation for the credit distribution function that better describes the tail behavior of credit loss would be more feasible.

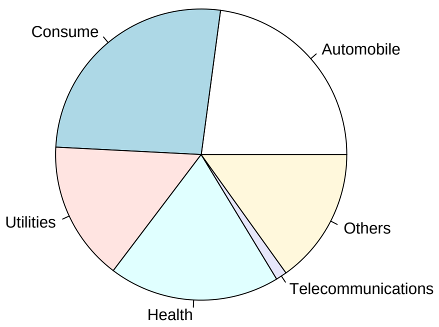

Figure 1: Industry sectors in the portfolio

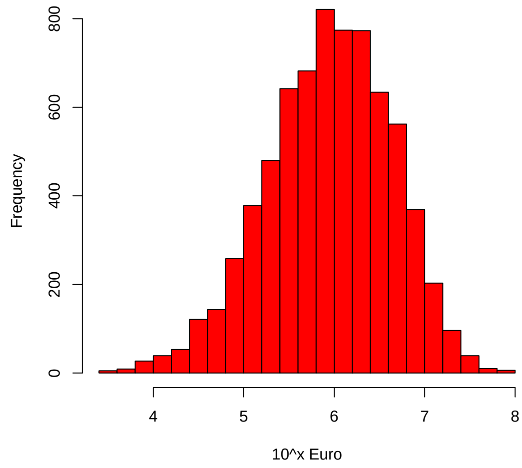

Figure 2: Histogram of the exposure sizes in the portfolio

6 Appendix

6.1 The Test Portfolios of Market and Credit Risk

The sample credit portfolio consists of 7124 loans with a total exposure of ca. 18 Billion Euro. The obligors are assigned to 7 different industry sectors such as car industry or telecommunication. The single exposures range between 3200 and 82 million Euro. The default probabilities are given with an average default probability of 2.08% and an exposure-weighted average default probability of 1.21%. Loss given default is set to 45%.

The standard deviation σ of the market portfolio, represented by the single random variable Z, is 180 million Euro.

In the normal factor approaches, dependence between the asset log returns $A_i, i = 1, \ldots, n$, of the obligors and the market loss variable Z is due to the systematic factors $Y_k, k = 1, \ldots, K$. The corresponding factor loadings β_{ik} and γ_k are estimated by Maximum-Likelihood factor analysis, see e.g. Fahrmeir et al. [6]. However, since we cannot observe the A_i directly, we assume that the dependence between the corresponding industry sector stock indices is similar and hence we use them for the estimation of the factor loadings.

For the market loss variable we use P/L figures as they typically occur in trading business. The data range is between February 2002 and March 2006.

6.2 Proofs

Proof of Theorem 3.1. First we calculate the covariance between $L^{(n)}$ and Z. Using $\mathbb{E}(Z) = 0$, expression (2.1), and the fact that η in (2.10) is independent from $Y_k, k = 1, \ldots, K$ (and thus from L_i), we can write

$$\operatorname{cov}(L^{(n)}, Z) = \mathbb{E}(Z L^{(n)}) = \sum_{i=1}^{n} e_i \, \mathbb{E}(Z L_i) = -\sigma \sum_{i=1}^{n} e_i \sum_{k=1}^{K} \gamma_k \, \mathbb{E}(Y_k L_i) \,. \tag{6.5}$$

To evaluate the expectation, conditioning with respect to $Y_k = y_k$ and using the law of iterated expectation yields

$$\begin{aligned}
\mathbb{E}(Y_k \, L_i) &= \mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K)) \\
&= \mathbb{E}\big(\mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K) \,|\, Y_k)\big) \\
&= \int_{-\infty}^{\infty} \mathbb{E}(Y_k \, L_i(Y_1, \ldots, Y_K) \,|\, Y_k = y_k) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} \mathbb{E}(y_k \, L_i(Y_1, \ldots, y_k, \ldots, Y_K)) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{E}(L_i(Y_1, \ldots, y_k, \ldots, Y_K)) \; d\Phi(y_k)
\end{aligned}$$

where Φ is the standard normal distribution function. Using $\mathbb{E}(L_i) = \mathbb{P}(L_i = 1)$, we have

$$\begin{aligned}
\mathbb{E}(Y_k \, L_i) &= \int_{-\infty}^{\infty} y_k \, \mathbb{P}(L_i(Y_1, \ldots, y_k, \ldots, Y_K) = 1) \; d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(\sum_{\substack{l=1 \\ l \neq k}}^{K} \beta_{il} \, Y_l + \beta_{ik} \, y_k + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \le D_i \right) d\Phi(y_k) \\
&= \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(\sum_{\substack{l=1 \\ l \neq k}}^{K} \beta_{il} \, Y_l + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \le D_i - \beta_{ik} \, y_k \right) d\Phi(y_k) \\
&=: \int_{-\infty}^{\infty} y_k \, \mathbb{P}\left(X \le D_i - \beta_{ik} \, y_k\right) d\Phi(y_k)
\end{aligned}$$

where X is normally distributed with variance $\operatorname{var}(X) = 1 - \beta_{ik}^2$. Hence, we finally obtain

$$\mathbb{E}(Y_k \, L_i) = \int_{-\infty}^{\infty} y_k \, \Phi\left(\frac{D_i - \beta_{ik} \, y_k}{\sqrt{1 - \beta_{ik}^2}} \right) d\Phi(y_k) \,.$$

The integral can be evaluated to

$$\mathbb{E}(Y_k \, L_i) = -\frac{\beta_{ik}}{\sqrt{2\pi}} \, e^{-\frac{D_i^2}{2}} \,, \tag{6.6}$$

which together with (6.5) yields

$$\begin{aligned}
\operatorname{cov}(L^{(n)}, Z) &= \frac{\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} \sum_{k=1}^{K} e_i \, \gamma_k \, \beta_{ik} \, e^{-\frac{D_i^2}{2}} \\
&= \frac{\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} e_i \, r_i \, e^{-\frac{D_i^2}{2}}
\end{aligned}$$

where we have introduced $r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^{K} \beta_{ik}\gamma_k$. With $\sqrt{\operatorname{var}(Z)} = \sigma$ and

$$\begin{aligned}
\operatorname{var}(L^{(n)}) &= \sum_{i,j=1}^{n} e_i \, e_j \operatorname{cov}(L_i, L_j) \\
&= \sum_{i,j=1}^{n} e_i \, e_j \left(\mathbb{E}(L_i L_j) - \mathbb{E}(L_i)\mathbb{E}(L_j)\right) \\
&= \sum_{i,j=1}^{n} e_i \, e_j \, (p_{ij} - p_i \, p_j) \,, \qquad (6.7)
\end{aligned}$$

where p_{ij} is the joint default probability (2.5), the assertion follows. □

Proof of Proposition 3.2. Since the obligor's exposures are assumed to be positive, $e_i \geq 0$, it follows from (3.2) that

$$|\operatorname{corr}(L^{(n)}, Z)| \leq \frac{\sum_i e_i \, |r_i| \exp\left(-\frac{1}{2}D_i^2\right)}{\sqrt{2\pi \sum_{ij} e_i \, e_j \, (p_{ij} - p_i \, p_j)}} \,.$$

¿From the Cauchy-Schwarz inequality together with $\sum_{k=1}^{K} \gamma_k^2 \leq 1$, it follows that

$$|r_i| = \left|\sum_{k=1}^{K} \beta_{ik}\gamma_k\right| \leq \left(\sum_{k=1}^{K} \beta_{ik}^2\right)^{1/2} \left(\sum_{k=1}^{K} \gamma_k^2\right)^{1/2} \leq \left(\sum_{k=1}^{K} \beta_{ik}^2\right)^{1/2} \leq 1 \,,$$

which completes the proof. □

Proof of Theorem 3.3.

(1) Using (2.1) and the law of iterated expectation, we obtain

$$\operatorname{cov}(L^{(n)}, Z) = \sum_{i=1}^{n} e_i \, \mathbb{E}(Z L_i) = \sum_{i=1}^{n} e_i \, \mathbb{E}\big(\mathbb{E}(Z L_i \,|\, W_L, W_Z)\big). \qquad (6.8)$$

Now, in the credit shock model of Definition 2.3 we have that for each loss variable L_i

$$\begin{aligned}
\mathbb{P}(L_i = 1) &= \mathbb{P}\left(W_L \sum_{k=1}^{K} \beta_{ik} \, Y_k + W_L \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \leq D_i\right) \\
&= \mathbb{P}\left(\sum_{k=1}^{K} \beta_{ik} \, Y_k + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2} \; \varepsilon_i \leq \frac{D_i}{W_L}\right) .
\end{aligned}$$

Hence, the shock factor model conditional on the shock variable W_L is equivalent to a normal factor model with adjusted default points $D_i^* := D_i/W_L$. Therefore, we obtain from (6.6) without any further calculation

$$\begin{aligned}
\mathbb{E}(ZL_i \,|\, W_L, W_Z) &= -\sigma \sum_{k=1}^{K} \gamma_k \, W_Z \, \mathbb{E}(Y_k L_i \,|\, W_L) \\
&= -\sigma \sum_{k=1}^{K} \gamma_k \, W_Z \left(-\frac{\beta_{ik}}{\sqrt{2\pi}} \, e^{-\frac{D_i^{*2}}{2}} \right) \\
&= \frac{\sigma \, r_i}{\sqrt{2\pi}} \, W_Z \, e^{-\frac{D_i^{*2}}{2}} ,
\end{aligned}$$

where $r_i = \sum_{k=1}^{K} \beta_{ik}\gamma_k$ for $i = 1, \ldots, n$. Integration over W_L and W_Z yields

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}(ZL_i \,|\, W_L, W_Z)\big) &= \frac{\sigma \, r_i}{\sqrt{2\pi}} \int_0^\infty W_Z \, dF_{\nu_Z}(s) \int_0^\infty e^{-\frac{D_i^{*2}}{2}} \, dF_{\nu_L}(s) \qquad (6.9) \\
&= \frac{\sigma \, r_i}{\sqrt{2\pi}} \, \mathbb{E}(W_Z) \int_0^\infty e^{-\frac{D_i^2}{2\nu_L} s} \, f_{\nu_L}(s) \, ds
\end{aligned}$$

where F_ν is the distribution function of a χ^2_ν distributed random variable with density $f_\nu(s)$. By substitution, we can perform the integration for $\nu_L > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\frac{D_i^2}{2\nu_L} s} \, f_{\nu_L}(s) \, ds &= \int_0^\infty \frac{2^{-\nu/2} \, s^{\nu/2-1}}{\Gamma\big(\frac{\nu}{2}\big)} \exp\left[-\left(1 + \frac{D_i^2}{\nu_L}\right) \frac{s}{2} \right] ds \\
&= \left(1 + \frac{D_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}} \int_0^\infty \frac{2^{-\nu/2} \, e^{-s/2} \, s^{\nu/2-1}}{\Gamma\big(\frac{\nu}{2}\big)} \, ds \\
&= \left(1 + \frac{D_i^2}{\nu_L}\right)^{-\frac{\nu_L}{2}} ,
\end{aligned}$$

so altogether

$$\mathbb{E}\big(\mathbb{E}(ZL_i \,|\, W_L, W_Z)\big) = \frac{\sigma \, r_i}{\sqrt{2\pi}} \sqrt{\frac{\nu_Z}{2}} \left(1 + \frac{D^2}{\nu_L}\right)^{\frac{-\nu_L}{2}} \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\big(\frac{\nu_Z}{2}\big)}, \qquad (6.10)$$

Now, plugging (6.10) into (6.8), and using (6.7) together with

$$\operatorname{var}(Z) = \left(\frac{\nu_Z}{\nu_Z - 2}\right) \sigma^2 ,$$

finally leads to

$$\operatorname{corr}(L^{(n)}, Z) = \sqrt{\frac{\nu_Z - 2}{2}} \, \frac{\Gamma\left(\frac{\nu_Z - 1}{2}\right)}{\Gamma\big(\frac{\nu_Z}{2}\big)} \, \frac{\sum_{i=1}^{n} e_i \, r_i \left(1 + \frac{D_i^2}{\nu_L}\right)^{\frac{-\nu_L}{2}}}{\sqrt{2\,\pi \operatorname{var}(L^{(n)})}} .$$

(2) Similarly to the case of independent shock variables, we are now conditioning on the single shock variable W. Instead of (6.9), we obtain for $\nu > 1$ by substitution

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}(ZL_i\,|\,W)\big) &= \frac{\sigma\, r_i}{\sqrt{2\pi}} \int_0^\infty W\, e^{-\frac{D_i^{*2}}{2}}\, dF_\nu(s) \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}} \int_0^\infty \sqrt{\frac{\nu}{s}}\, e^{-\frac{D_i^2}{2\nu} s}\, f_\nu(s)\, ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}} \int_0^\infty \frac{2^{-\nu/2}\,\nu^{1/2}}{\Gamma(\nu/2)} s^{\frac{\nu-1}{2}-1} \exp\left[-\left(\frac{D_i^2}{\nu}+1\right)\frac{s}{2}\right] ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}} \int_0^\infty \frac{2^{-\nu/2}\,\nu^{1/2}}{\Gamma(\nu/2)} \left(\frac{D_i^2}{\nu}+1\right)^{-\frac{\nu-1}{2}} s^{\frac{\nu-1}{2}-1} \exp\left[-\frac{s}{2}\right] ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}} \frac{\nu^{1/2}\,2^{-1/2}}{\Gamma(\nu/2)} \left(\frac{D_i^2}{\nu}+1\right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu-1}{2}\right) \int_0^\infty \frac{2^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-1}{2}\right)} s^{\frac{\nu-1}{2}-1} e^{-\frac{s}{2}} ds \\
&= \frac{\sigma\, r_i}{\sqrt{2\pi}} \frac{\nu^{\frac{\nu}{2}}\,(D_i^2+\nu)^{\frac{1-\nu}{2}}\,\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{\nu}{2}\right)}\,, \qquad (6.11)
\end{aligned}$$

which finally completes the proof. □

Proof of Proposition 3.4. See proof of Proposition 3.2.

Proof of Proposition 5.1. An important characteristic of copulas is their invariance under monotonously increasing transformations. Since the portfolio loss L in the LHP approximation (see (4.2)) is a monotonously increasing function of $-\tilde{Y}$, it follows that L and Z have the same copula as $-\tilde{Y}$ and Z. From the one-factor representation of market risk (4.5) it follows that these variables are bivariate standard normally distributed with correlation

$$\operatorname{corr}(-\tilde{Y}, Z) = \tilde{\gamma}\,.$$

Hence, also L and Z are linked by a Gaussian copula with correlation parameter $\tilde{\gamma}$. The second part of the assertion follows directly from (4.7) and (4.8) together with $r = \sqrt{\rho}\,\tilde{\gamma}$.

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